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On Properties of Matroid Connectivity

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Abstract

Highly connected matroids are consistently useful in the analysis of matroid structure. Round matroids, in particular, were instrumental in the proof of Rota's conjecture. Chapter 2 concerns a class of matroids with similar properties to those of round matroids. We provide many useful characterizations of these matroids, and determine explicitly their regular members.

Tutte proved that a 3-connected matroid with every element in a 3-element circuit and a 3-element cocircuit is either a whirl or the cycle matroid of a wheel. This result led to the proof of the 3-connected splitter theorem. More recently, Miller proved that matroids of sufficient size having every pair of elements in a 4-element circuit and a 4-element cocircuit are spikes. This observation simplifies the proof of Rota's conjecture for $GF(4)$. In Chapters 3 and 4, we investigate matroids having similar restrictions on their small circuits and cocircuits. The main result of each of these chapters is a complete characterization of the matroids therein.

Chapter 1
Introduction

Throughout this dissertation, we follow the conventions of Oxley [5], and we assume that the reader is familiar with the basic concepts of matroid theory, including rank, duality, minors, and connectivity.

We also assume that the reader has an understanding of some basic graph theory. One class of graphs that will arise frequently are the wheel graphs. A *wheel*, $\mathcal{W}_n$, is a graph consisting of a cycle of length n, called the *rim*, and one additional vertex that is adjacent to each other vertex. A *whirl*, $\mathcal{W}^n$, is a matroid obtained by relaxing the rim of $M(\mathcal{W}_n)$.

Chapter 2
Unbreakable Matroids

2.1 Preliminaries and Equivalent Characterizations

This chapter is devoted to the study of matroids that remain connected upon contracting any flat. Specifically, we call a matroid M *unbreakable* if M is connected and, for every flat F of M, the matroid M/F is also connected. One attractive feature of unbreakable matroids is their many useful equivalent characterizations, presented in Theorem 2.1. One of these characterizations is defined in terms of the *local connectivity* $\sqcap_M(S_1, S_2)$, or $\sqcap(S_1, S_2)$, of two subsets S_1 and S_2 of M, defined by

$$\sqcap(S_1, S_2) := r(S_1) + r(S_2) - r(S_1 \cup S_2). \tag{2.1.1}$$

Two subsets are called *skew* if their local connectivity is 0, and if those subsets partition the ground set they form a 1-*separation* of M. One of the characterizations we show is that M is unbreakable if, and only if, M^* has no skew circuits. We say $N = M/e$ is a *parallel deletion* of M if e is in a 2-circuit of M. We say N is a *parallel minor* of M if N can be obtained from M by a sequence of contractions and parallel deletions. Another characterization shows that M is unbreakable if and only if M does not have $U_{2,2}$ as a parallel minor. Recall that the simplification of a matroid M, denoted $\mathrm{si}(M)$, is the matroid obtained from M by deleting all loops and deleting all but one element from each parallel class.

Theorem 2.1. *The following statements are equivalent for a matroid M.*

(i) *M is unbreakable.*

(ii) *M^* has no skew circuits.*

(iii) *Every rank-$(r-2)$ flat of M is contained in at least three hyperplanes.*

(iv) *For all* $X \subseteq E(M)$, $\mathrm{si}(M/X) \ncong U_{2,2}$, *for all* $X \subseteq E(M)$.

(v) M/F *is unbreakable for all rank-1 flats* F *of* M.

(vi) *For every partition* (X, Y) *of* $E(M)$ *with* $X, Y \neq \emptyset$, *if* X' *is a flat that is properly contained in* X *and* $Y' \subseteq Y$, *then* $\sqcap(X', Y') < \sqcap(X, Y)$.

Proof. The structure of the proof is as follows: we shall show that **(i)** implies **(iv)**, that **(iv)** implies **(iii)**, that **(iii)** implies **(ii)**, and that **(ii)** implies **(i)**. Then we shall show the equivalence of **(i)** and **(v)**, and lastly the equivalence of **(i)** and **(vi)**.

To show that **(i)** implies **(iv)**, let M be unbreakable. A subset $X \subseteq E(M)$ such that $\mathrm{si}(M/X) \cong U_{2,2}$ cannot exist, since $\mathrm{si}(M/\mathrm{cl}(X)) \cong \mathrm{si}(M/X)$, and $\mathrm{si}(M/\mathrm{cl}(X))$ is connected since M is unbreakable. Therefore **(i)** implies **(iv)**.

We show that **(iv)** implies **(iii)**, by proving the contrapositive. Suppose F is a rank-$(r-2)$ flat of M contained in exactly two hyperplanes H_1 and H_2. Then $F = H_1 \cap H_2$ and $r(M/F) = 2$. Further, M/F must consist of two disjoint rank-1 flats. The only possibility, then, is that $\mathrm{si}(M/F) = U_{2,2}$. We conclude that **(iv)** implies **(iii)**.

Now suppose **(iii)** holds. To show that **(ii)** holds, let D_1 and D_2 be cocircuits of M, and let $H_i = E(M) - D_i$ for each $i \in \{1, 2\}$. Then

$$\begin{aligned}
\sqcap_{M^*}(D_1, D_2) &= r_{M^*}(D_1) + r_{M^*}(D_2) - r_{M^*}(D_1 \cup D_2) \\
&= |D_1| + |D_2| - 2 - [r_M(E(M) - (D_1 \cup D_2)) + |D_1 \cup D_2| - r(M)] \\
&= |D_1 \cap D_2| - 2 - r_M(H_1 \cap H_2) + r_M(M) \\
&\geq |D_1 \cap D_2| - 2 - (r(M) - 2) + r_M(M) \\
&= |D_1 \cap D_2|.
\end{aligned}$$

Since equality holds only when $r_M(H_1 \cap H_2) = r(M) - 2$, we need only argue that, in this case, $|D_1 \cap D_2| \neq 0$. Let $F = H_1 \cap H_2$. Then F is contained in at least three distinct hyperplanes by assumption. There must be an element $e \in E(M) - F$ such that $\mathrm{cl}(e \cup F) \neq H_i$ for $i \in \{1, 2\}$. Therefore, $|D_1 \cap D_2| = |E(M) - (H_1 \cup H_2)| \geq 1$. Thus **(iii)** implies **(ii)**.

Next, suppose that **(ii)** holds, but **(i)** does not. Then M has a flat F such that M/F is not connected. Now, for $n = r(M) - r(F)$, there are hyperplanes $H_1, H_2, \ldots, H_n$ of M such that $F = \bigcap_{i=1}^{n} H_i$. Note that $n \neq 1$, as M/H is a rank-one loopless matroid and so is connected. Hence, $n \geq 2$. Then, if we let D_i be the corresponding cocircuit complement of each H_i, we get

$$\begin{aligned}
M/F &= M/[H_1 \cap H_2 \cap \cdots \cap H_n] \\
&= M/[(E(M) - D_1) \cap (E(M) - D_2) \cap \cdots \cap (E(M) - D_n)] \\
&= M/[E(M) - (D_1 \cup D_2 \cup \cdots \cup D_n)] \\
&= M^*\backslash[E(M) - (D_1 \cup D_2 \cup \cdots \cup D_n)] \\
&= M^*|(D_1 \cup D_2 \cup \cdots \cup D_n).
\end{aligned}$$

Since M/F is not connected, we know $M' = M^*|(D_1 \cup D_2 \cup \ldots \cup D_n)$ is not connected. Hence, there must be some partition (S, T) of M' such that $\lambda_{M'}(S, T) = 0$. This implies that each D_i is either contained in S or contained in T. Therefore, there must be cocircuits D_i and D_j for some $\{i, j\} \subseteq \{1, 2, \ldots n\}$ such that $\sqcap_{M^*}(D_i, D_j) = 0$, a contradiction. Thus **(ii)** implies **(i)**.

To show that **(i)** implies **(v)**, assume M is unbreakable, and suppose there is a rank-1 flat F of M such that M/F is not unbreakable. Then there must be some flat G of M/F such that $(M/F)/G$ is not connected. This is a contradiction, since $G \cup F$ is a flat of M, and M is unbreakable by assumption. Therefore **(i)** implies **(v)**.

Now assume **(v)** holds. We shall show that M is unbreakable. Let F be a flat of M, and let $e \in F$. Since F is closed, we know $\mathrm{cl}(e) \subseteq F$. Then $M/F = M/(\mathrm{cl}(e) \bigcup (F - \mathrm{cl}(e))) =$

$(M/\mathrm{cl}(e))/(F - \mathrm{cl}(e))$, which is connected since $M/\mathrm{cl}(e)$ is unbreakable and has $F - \mathrm{cl}(e)$ as a flat. Therefore M is unbreakable, and **(v)** implies **(i)**.

Next, we show that **(i)** implies **(vi)**. Assume M is unbreakable, and suppose (X, Y) partitions $E(M)$, neither X nor Y are empty, X' is a flat properly contained in X, and $Y' \subseteq Y$. Suppose $\sqcap(X', Y') = \sqcap(X, Y)$. Then $\sqcap(X', Y) = \sqcap(X, Y)$. Therefore $r(X') = r(X) - r(X \cup Y) + r(X' \cup Y)$. Now we consider $M' = M/X'$. Then

$$\begin{aligned}
\sqcap_{M'}(X - X', Y) &= r_{M'}(X - X') + r_{M'}(Y) - r_{M'}((X - X') \cup Y) \\
&= r_M(X) - r_M(X') + r_{M'}(Y) - (r(M) - r_M(X')) \\
&= r_M(X) + r_M(Y \cup X') - r_M(X') - r(M) \\
&= 0.
\end{aligned}$$

Thus, the contraction of X' yields a matroid that is not connected, a contradiction. Therefore **(i)** implies **(vi)**.

Now assume **(vi)** holds, but **(i)** does not. Then M has a flat F such that M/F is not connected. Let (X_F, Y_F) be a 1-separation of M/F. Consider $(X_F \cup F, Y_F)$, a partition of $E(M)$. We will show that $\sqcap(X_F \cup F, Y_F) = \sqcap(F, Y_F)$. Observe that

$$\begin{aligned}
\sqcap_{M/F}(X_F, Y_F) &= r_{M/F}(X_F) + r_{M/F}(Y_F) - r_{M/F}(X_F \cup Y_F) \\
&= r_M(X_F \cup F) - r_M(F) + r_M(Y_F \cup F) - r_M(F) - r_M(M) + r_M(F) \\
&= r_M(X_F \cup F) + r_M(Y_F \cup F) - r_M(M) - r_M(F) \\
&= 0.
\end{aligned}$$

Thus

$$r_M(X_F \cup F) - r(M) = r_M(F) - r_M(Y_F \cup F),$$

and therefore

$$\begin{aligned}
\sqcap_M(X_F \cup F, Y_F) &= r_M(X_F \cup F) - r(M) + r_M(Y_F) \\
&= r_M(F) - r_M(Y_F \cup F) + r_M(Y_F) \\
&= \sqcap_M(F, Y_F).
\end{aligned}$$

As this contradicts **(vi)**, we deduce that **(vi)** implies **(i)**. We conclude that the theorem holds.

$\square$

The following corollary is an immediate consequence of part **(v)** of the last theorem.

Corollary 2.2. *A loopless parallel minor of an unbreakable matroid is unbreakable.*

To close this section, we note the similarity between unbreakable matroids and round matroids. A matroid is called *round* if each of its cocircuits is spanning. Round matroids and unbreakable matroids have related equivalent characterizations, as seen by comparing the following theorem to Theorem 2.1. This yields an immediate corollary that all round matroids are unbreakable.

Theorem 2.3. *The following statements are equivalent for a matroid M:*

(i) *M is round.*

(ii) *M has no disjoint cocircuits.*

(iii) *M cannot be written as the union of two proper flats.*

(iv) *Every cocircuit of M is spanning.*

Corollary 2.4. *Let M be a matroid. If M is round, then M is unbreakable.*

2.2 Classifying Unbreakable Regular Matroids

In order to determine the unbreakable regular matroids, we will first find the unbreakable graphic and cographic matroids, and then apply Seymour's decomposition theorem for regular matroids.

Before we begin classifying these matroids, we need the following preliminary lemma.

Lemma 2.5. *If M is an unbreakable matroid and N is a matroid such that si$(N) \cong M$, then N is unbreakable.*

Proof. Let M' be such that $\mathrm{si}(M') = M$, and $M' \cong N$. For any flat F of M, we have $\mathrm{si}(M'/\mathrm{cl}_{M'}(F)) = M/F$ is connected. Therefore M' is unbreakable, since every flat of M' is the closure in M' of a flat of M. Thus N is unbreakable. $\square$

We will also use Tutte's characterization of graphs that are 2-connected but not 3-connected, called *generalized cycles*. Such a graph G can be expressed in *parts* $G_1, G_2, \ldots, G_n$ such that $n \geq 2$, each G_i is connected, their edge sets partition $E(G)$, each G_i shares exactly two vertices (called *contact vertices*) with $\bigcup_{j \neq i} G_j$, and if each G_i is replaced by an edge joining its contact vertices, the resulting graph is a cycle.

It is not difficult to see that the cycle matroids of the graphs C_n and K_n are unbreakable for all $n > 0$. The following proposition shows that these are essentially the only unbreakable graphic matroids.

Proposition 2.6. *A graphic matroid M is unbreakable if, and only if, for some $n > 0$, either si$(M) \cong M(C_n)$ or si$(M) \cong M(K_n)$.*

Proof. Let M be a graphic matroid such that $\mathrm{si}(M)$ is isomorphic to $M(C_n)$ or $M(K_n)$. If F is a rank-k flat of M, then $\mathrm{si}(M/F)$ is isomorphic to $M(C_{n-k})$ or $M(K_{n-k})$, respectively. As each of the last two matroids is connected, M is unbreakable.

Now, suppose M is an unbreakable graphic matroid, and let G be a connected graph such that $M(G) \cong M$. If $|V(G)| < 3$, then $\mathrm{si}(M) \cong M(K_n)$ for $n = |V(G)|$. Hence, we may assume that $|V(G)| \geq 3$.

Suppose first that G is 3-connected and $\mathrm{si}(M) \not\cong M(K_n)$. Then there are two non-adjacent vertices in G, say v_1 and v_2. Then $G \backslash \{v_1, v_2\}$ is connected, and $\mathrm{si}(M(G/E(G \backslash \{v_1, v_2\}))) \cong U_{2,2}$. Thus M is not unbreakable.

We may now suppose that G is not 3-connected. Assume that $\mathrm{si}(M) \not\cong M(C_n)$. As M is connected, G must be 2-connected. Therefore G is a generalized cycle with parts $G_1, G_2, \ldots, G_n$. One part, say G_v, must contain a vertex v such that, if $\{u, w\}$ are the contact vertices of G_v, there is a path from u to w not containing v. Let v' be a vertex not in $V(G_v)$. Let S be the set of all edges not incident with v or v'. Then $\mathrm{si}(M(G/S)) \cong U_{2,2}$. Thus M is not unbreakable. $\square$

Concerning unbreakable cographic matroids, we can approach the problem using Theorem 2.1**(ii)** by considering graphic matroids with no skew circuits. Skew circuits appear in a graph as cycles that share at most one vertex. The following is a theorem of Dirac [1] that determines all 3-connected simple graphs with no two vertex-disjoint cycles. The graphs $K'_{3,p}$, $K''_{3,p}$, and $K'''_{3,p}$ denote $K_{3,p}$ with one, two, and three additional edges between the vertices of the vertex class of size 3.

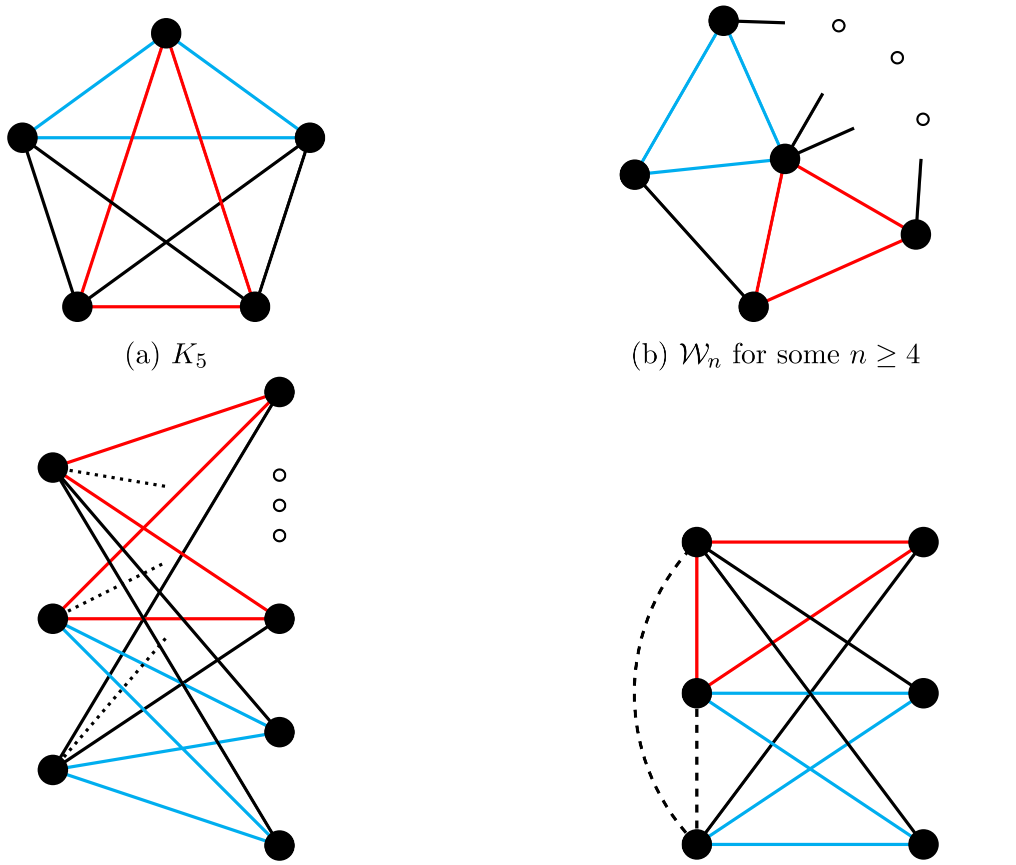
Theorem 2.7. *Every 3-connected graph with no two vertex-disjoint cycles is one of the following graphs:*

$\mathcal{W}_k$ $(k \geq 4)$, K_5, $K_5 \backslash e$, $K_{3,p}$, $K'_{3,p}$, $K''_{3,p}$, $K'''_{3,p}$ $(p \geq 3)$.

The 3-connected unbreakable cographic matroids must must form a subset of the bond matroids of the graphs in the previous theorem. Using this fact, we can find all the unbreakable cographic matroids. It is sufficient to determine those cographic unbreakable matroids that are not also graphic.

Proposition 2.8. *Let M be a matroid that is cographic but not graphic. Then M is unbreakable if, and only if, $si(M) \cong M^*(K_{3,3})$.*

Proof. By Theorem 2.1 **(ii)**, the cographic unbreakable matroids are all $M \cong M^*(G)$ such that $M(G)$ has no skew circuits; that is, all cycles of G must share at least two vertices. Therefore, if G is 3-connected, then G must be a graph from the list in Theorem 2.7. The only graph on this list in which all cycles share at least two vertices is $K_{3,3}$. See Figure 2.1 for a demonstration of this fact.



(a) K_5

(b) $\mathcal{W}_n$ for some $n \geq 4$

(c) $K_{3,n}$ for some $n \geq 4$

(d) $K'_{3,3}$, with possible additional edges

Figure 2.1: The graphs from Theorem 2.7 having two edge-disjoint cycles.

Now suppose G is not 3-connected. Then G must be 2-connected, and is therefore a generalized cycle with parts $G_1, G_2, \ldots, G_n$. At least one part of G, say G_k, must be non-planar. We may assume G_k is chosen to have no pendant edges, and is otherwise maximal.

No part of G besides G_k can contain a cycle; otherwise $M(G)$ would have skew circuits, which would contradict Theorem 2.1**(ii)**. Hence, G is isomorphic to a large cycle where one edge is replaced by the non-planar graph G_k. By a repeated application of Theorem 2.1 **(v)**, we may contract all the edges in $E(G) - E(G_k)$ and maintain unbreakability. Hence, $M(G_k)$ is unbreakable. Therefore $G_k \cong K_{3,3}$. Let $\{u, v\}$ be the contact vertices of G_k. Then we can find both a path from u to v and a cycle, C, in G_k such that they share at most one vertex, as demonstrated in Figure 2.2. Such a path forms a cycle with $E(G) - E(G_k)$, and this cycle is skew with C, a contradiction. □

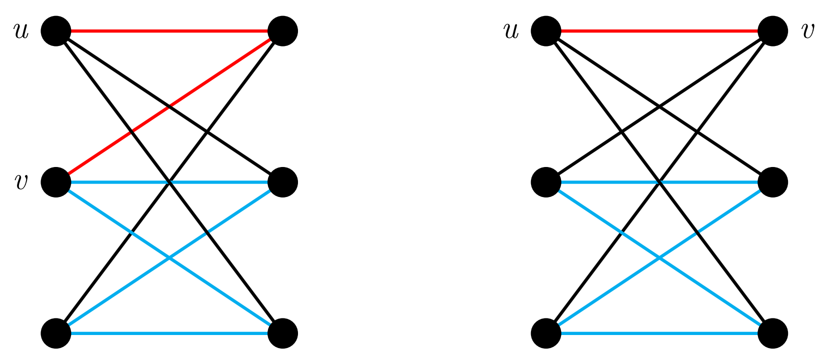


Figure 2.2: A path from u to v and a cycle sharing at most one vertex with it.

Using the previous two propositions, along with Seymour's decomposition theorem for regular matroids [6] restated here, we will be able to find all unbreakable regular matroids. Recall that R_{10} is the unique regular matroid on ten elements that is neither graphic nor cographic.

Theorem 2.9. *A regular matroid M can be constructed using* 1-, 2-, *and* 3-*sums of matroids that are either graphic, cographic, or isomorphic to R_{10}, and each matroid used in this construction is isomorphic to a minor of M.*

We will need a few preliminary lemmas before we prove the main result of this section. We call an element of a matroid *free* if it is contained in no non-spanning circuits. The first two lemmas concern the 2-sums of unbreakable matroids, requiring that the basepoint p of a 2-sum be free in both matroids in order to maintain unbreakability.

Lemma 2.10. *If a matroid M contains a free element, then M is unbreakable.*

Proof. Let M be a matroid with a free element p. Suppose M is not unbreakable. Then M has a flat F such that M/F is not connected. Therefore there are elements e_1 and e_2 in M/F such that there is no circuit containing both. Observe that e_1 and e_2 are not loops nor are they parallel, as any loops are contained in F, so parallel elements form a circuit. Let I_F be a maximal independent set in F. Then $r_M(I_F \cup e_1 \cup e_2) = r_M(I_F) + 2$, and $I_F \cup e_1 \cup e_2$ is independent. Let B_F be a basis of M containing $I_F \cup e_1 \cup e_2$. Then $B_F \cup p$ is a circuit, C_F, of M such that $r(F) = |C_F \cap F|$. Hence,

$$r_{M/F}(C_F - F) = r_M(C_F \cup F) - r_M(F) = r(M) - r_M(F),$$

and

$$|C_F - F| = |C_F| - |C_F \cap F| = r(M) + 1 - r_M(F).$$

Therefore $r_{M/F}(C_F - F) = |C_F - F| - 1$, and $C_F - F$ is a circuit of M/F containing both e_1 and e_2, a contradiction. Thus, M is unbreakable.

□

Lemma 2.11. *The matroid $M \cong (M_1, p) \bigoplus_2 (M_2, p)$ is unbreakable if, and only if, p is a free element in both M_1 and M_2.*

Proof. Suppose $(M_1, p) \bigoplus_2 (M_2, p)$ is unbreakable, but p is not a free element of M_1. Let C be a non-spanning circuit of M_1 containing p, and let $F = \mathrm{cl}(C)$. By the definition of 2-sum, the only circuits of M containing elements from both $E(M_1)$ and $E(M_2)$ are those of the form $(C_1 \cup C_2) - p$, where C_1 and C_2 are circuits of M_1 and M_2, respectively, that contain p. Therefore, in M/F, there are no circuits containing elements from both $E(M_1) - F$ and $E(M_2)$; that is, M/F is not connected. This is a contradiction.

Now suppose that p is free in both M_1 and M_2. By Lemma 2.10, both M_1 and M_2 are unbreakable. If M is not unbreakable, then there is a flat F of M, such that M/F is not connected. Note that F cannot be contained in either of M_1 or M_2. Therefore, $F = F_1 \cup F_2$, where each F_i is a flat of M_i. There must, then, be two elements e_1 and e_2 of M/F that are not in a circuit together. Note that neither element is a loop. Suppose $e_1 \in E(M_1)$. Then $e_2 \in E(M_2)$ since M_1 is unbreakable. As in the previous lemma, we can form a spanning circuit C_i containing p in each M_i such that $|C_i \cap F_i| = r_{M_i}(F_i)$ and $e_i \in C_i$. Then $C = (C_1 \cup C_2) - \{p\}$ is a circuit of M, such that $|C - F| = r_{M/F}(C - F) + 1$. Therefore $C - F$ is a circuit of M/F containing both e_1 and e_2, a contradiction. Thus, M is unbreakable.

□

The two lemmas that follow describe the utility of the 3-sum. Further information and proofs of these lemmas can be found under Proposition 9.3.5 and Proposition 11.4.14 in [5]. The proof given for the former result actually shows that each of M_1 and M_2 is a parallel minor of M, and the statement here reflects this. The latter result appears as a property of the generalized parallel connection in [5]; however, we restate it here in terms of 3-sums.

Lemma 2.12. *If a 3-connected matroid M is the 3-sum of binary matroids M_1 and M_2, then M has parallel minors that are isomorphic to each of M_1 and M_2.*

Lemma 2.13. *Let M_1 and M_2 be binary matroids with $E(M_1) \cap E(M_2) = T$, where $M_1|T = M_2|T$ is a triangle. If $e \in E(M_1) - \mathrm{cl}_1(T)$, then $(M_1 \bigoplus_3 M_2)/e = (M_1/e) \bigoplus_3 M_2$.*

The following is the main result of this chapter.

Theorem 2.14. *A regular matroid M is unbreakable if, and only if, $si(M)$ is isomorphic to one of $M(C_n), M(K_n), M^*(K_{3,3})$, or R_{10}.*

Proof. By Propositions 2.6 and 2.8, we only need to show that R_{10} is unbreakable in order to prove that each listed matroid is unbreakable. We know $r(R_{10}) = 5$ and the smallest circuit of R_{10} has 4 elements. If C_1 and C_2 are circuits of R_{10}, then $\sqcap(C_1, C_2) = r(C_1) + r(C_2) -$

$r(C_1 \cap C_2) \geq 3 + 3 - 5 = 1$. Therefore R_{10} has no skew circuits and, since R_{10} is self-dual, by Theorem 2.1**(ii)**, it is unbreakable.

Now let M be an unbreakable regular matroid. We may assume that M is simple. Suppose M is not isomorphic to any of the matroids listed above and that $|E(M)|$ is a minimum among such matroids. By Theorem 2.9, M can be obtained by 1-, 2-, and 3-sums of graphic matroids, cographic matroids, and R_{10}. Clearly M cannot be the result of a 1-sum.

Suppose M can be decomposed via a 2-sum, say $M = (M_1, p) \bigoplus_2 (M_2, p)$. By Lemma 2.11, each M_1 and M_2 must have p as a free element. By Lemma 2.10, having a free element implies that a matroid is unbreakable. Therefore, each M_i must be a member of the previously determined list of unbreakable matroids. However, the only member from that list having a free element is M such that $\mathrm{si}(M) \cong C_n$, and the 2-sum of circuits simply yields a circuit. Thus M does not have a 2-separation. Hence, M is 3-connected.

Finally, suppose $M \cong M_1 \bigoplus_3 M_2$. By Lemma 2.12, each of M_1 and M_2 is isomorphic to a loopless parallel minor of M and so, by Corollary 2.2, is unbreakable. Therefore, each M_i must be one of the previously identified unbreakable matroids and must contain a triangle. Hence, the only candidates are M_i such that $\mathrm{si}(M_i) \cong M(K_n)$ or $\mathrm{si}(M_i) \cong M^*(K_{3,3})$, when $n \geq 4$. As K_4 is a minor of each K_n when $n \geq 4$, it suffices to consider $M(K_4) \bigoplus_3 M(K_4)$, $M(K_4) \bigoplus_3 M^*(K_{3,3})$, and $M^*(K_{3,3}) \bigoplus_3 M^*(K_{3,3})$. Here we abusing notation since the definition of 3-sum requires that each part have at least seven elements. By $M(K_4) \bigoplus_3 M(K_4)$ we mean the graphic matroid $M(G)$ with G obtained by identifying the edges of a triangle in two copies of K_4 and then deleting those edges, and by $M(K_4) \bigoplus_3 M^*(K_{3,3})$ we mean the matroid whose geometric representation is seen as the first in the sequence in Figure 2.4. Note that $M(K_4) \bigoplus_3 M(K_4)$ is graphic and differs from both C_n and K_n, and so, by Theorem 2.6, is not unbreakable. It is easily checked that the others contain a flat whose contraction produces a matroid that is not connected, as demonstrated in Figures 2.3 and 2.4. The matroids $M^*(K_{3,3})$ has rank 4, so $M^*(K_{3,3}) \bigoplus_3 M^*(K_{3,3})$ has rank 5. Figure 2.3

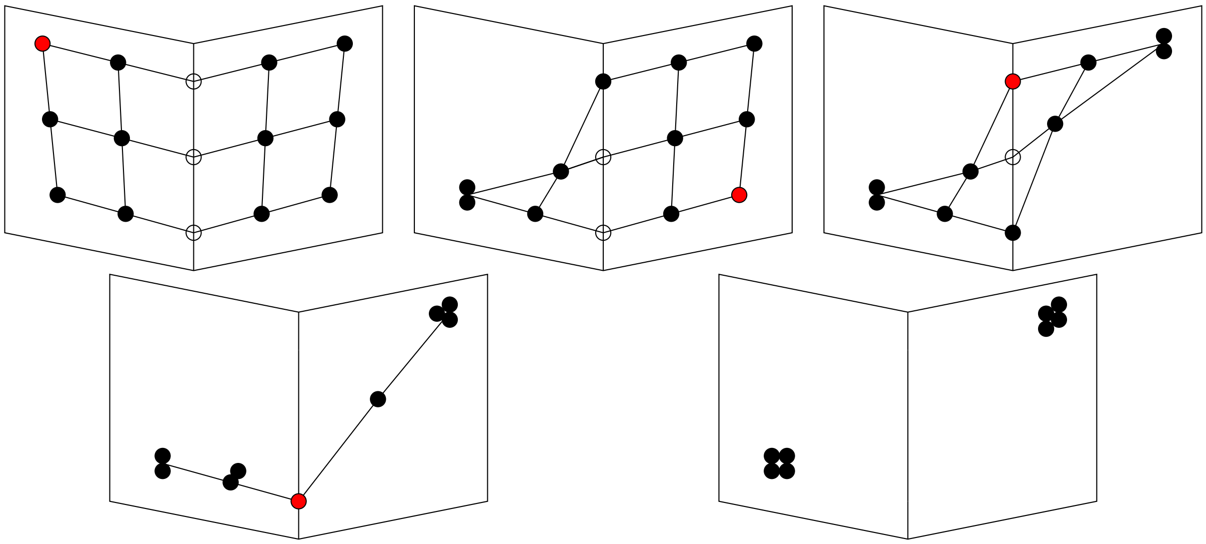


Figure 2.3: A sequence of contractions that disconnects $M(K_{3,3}) \bigoplus_3 M(K_{3,3})$.

gives a representation of the last matroid obtained by combining geometric representations of two copies of $M^*(K_{3,3})$. Figure 2.4 begins with a representation $M^*(K_{3,3}) \bigoplus_3 M(K_4)$, followed by graph pictures. Therefore there are no additional unbreakable regular matroids formed via 3-sum, and our list is complete.

□

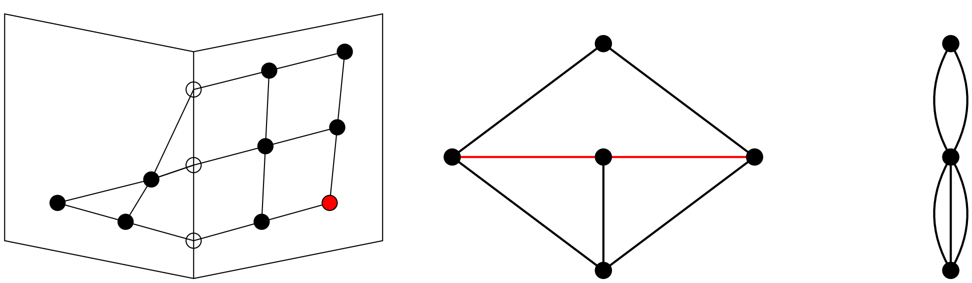


Figure 2.4: A sequence of contractions that disconnects $M(K_{3,3}) \bigoplus_3 M(K_4)$.

2.3 Unbreakable Representable Matroids

A natural next step in classifying unbreakable matroids is to examine the unbreakable representable matroids. This is, as expected, more difficult than in the previous cases, and, as our results indicate, the variety of unbreakable matroids increases significantly as we begin to consider larger classes of representable matroids. It is straightforward to determine that $PG(r-1,q)$ and $AG(r-1,q)$, with $r \geq 1$ and $q \geq 2$, are among the unbreakable representable matroids. Unlike in the regular case, these examples are not minimal; that is, the deletion of elements from either of these matroids may produce another unbreakable matroid. To make this notion more precise, we have the following results.

Theorem 2.15. *Let $S \subseteq E(PG(r-1,q))$. If $|S| \leq q^{r-1} - q^{r-2} - 1$, then $PG(r-1,q)\backslash S$ is unbreakable.*

Proof. For every $q \geq 3$, $PG(1,q)$ is isomorphic to the $(q+1)$-point line $U_{2,q}$. By Theorem 2.1.**(iv)**, we know that a matroid is unbreakable as long as it has no contraction minor whose simplification is isomorphic to $U_{2,2}$. Also, for $e \in E(PG(r-1,q))$, the matroid $PG(r-1,q)/e$ is isomorphic to $PG(r-2,q)$ with each of its elements replaced by q elements in parallel. If $\{e_1, e_2, \ldots, e_{r-2}\}$ is an independent set in $PG(r-1,q)$, then $PG(r-1,q)/\{e_1, e_2, \ldots, e_{r-2}\}$ is isomorphic to $PG(1,q)$ with every element replaced by q^{r-2} elements in parallel. We can delete $(q-2)q^{r-2} + q^{r-2} - 1$ elements from this matroid without the possibility of reducing its rank to 2 if we delete all elements in all but three parallel classes, and then $q^{r-2} - 1$ elements from one of the remaining parallel classes. Therefore, we can delete $q^{r-1} - q^{r-2} - 1$ elements from $PG(r-1,q)$ without creating a contraction minor whose simplification is $U_{2,2}$. Thus the desired result holds. □

To see that the above bound and those that follow are tight, note that deleting $q^{r-1} - q^{r-2}$ elements is enough to remove all elements except those in two parallel classes of $PG(r-1,q)/\{e_1, e_2, \ldots, e_{r-2}\}$, where $\{e_1, e_2, \ldots, e_{r-2}\}$ is an independent set. That is, we can find a set $S \subseteq E(PG(r-1,q))$ with $|S| = q^{r-1} - q^{r-2}$ such that $\mathrm{si}(PG(r-1,q)/S) \cong U_{2,2}$.

Corollary 2.16. *Let $S \subseteq E(AG(r-1,q))$. If $|S| \leq q^{r-2} - q^{r-3} - 1$, then $AG(r-1,q)\backslash S$ is unbreakable.*

Proof. The proof of this is nearly identical to the previous, once we note that $AG(r-1,q)/e \cong PG(r-2,q)$. We omit the details. $\square$

In the binary case, we have the following easy corollary, with which we close the chapter.

Corollary 2.17. *If M is a simple rank-r binary matroid having at least $2^r - 2^{r-1} + 2^{r-2}$ elements, then M is unbreakable.*

Chapter 3
Many Triads and 4-circuits

3.1 Introduction and Preliminaries

The study of matroids with many small circuits and cocircuits begins with Tutte's well-known Wheels-and-Whirls Theorem [7]. The result was originally stated in terms of *essential* elements of a 3-connected matroid; that is, elements that destroy the 3-connectedness of the matroid both on deletion and on contraction. We present it here in terms of 3-circuits and 3-cocircuits.

Theorem 3.1. *Suppose M is a non-empty 3-connected matroid. Every element of M is in both a 3-circuit and a 3-cocircuit if and only if M has rank at least three and is isomorphic to a wheel or a whirl.*

This result has been instrumental in the analysis of 3-connected matroids. Seymour's Splitter Theorem 2.9 is a well-known extension of the last theorem. More recently, Miller [4] proved the following result, which requires conditions similar to those in Tutte's theorem. A *spike* is a rank-r matroid M whose ground set E is $\{x_1, y_1, x_2, y_2, \ldots, x_r, y_r\}$, and whose circuits consist of the following sets:

(i) all sets of the form $\{x_i, y_i, x_j, y_j\}$ with $1 \leq i < j \leq r$,

(ii) a subset of $\{\{z_1, z_2, \ldots, z_r\} \ : \ z_i \in \{x_i, y_i\} \ \forall \ i\}$ such that no two members of this subset have more than $r-2$ common elements, and

(iii) all $(r+1)$-element subsets of E that contain none of the sets in (i) or (ii).

It should be noted that what we have just defined to be a spike is also known as a tipless spike.

Theorem 3.2. *Let M be a matroid in which every pair of elements belongs to a 4-circuit and a 4-cocircuit. If $|E(M)| \geq 13$, then M is a spike.*

In this chapter, we continue along a similar line of inquiry by investigating matroids M having the following property:

3.3. A matroid M has property $(P1)$ if, for all $\{e, f\} \in E(M)$, we have:

(i) there exists some 4-circuit $C \in \mathcal{C}(M)$ such that $\{e, f\} \in C$,

(ii) there exists some 3-cocircuit $D \in \mathcal{C}(M^*)$ such that $e \in D$, and

(iii) M is 3-connected.

We will assume throughout this chapter that M has property $(P1)$, and will proceed to determine all such matroids. In order to achieve this, we must first make several observations about the structural consequences of $(P1)$. These lemmas will allow us to determine explicitly the possibilities for M when $|E(M)| \leq 8$. We conclude by showing that, when M is sufficiently large, it belongs to a familiar family of matroids; namely $M \cong M(K_{3,n})$ for some $n \geq 3$. Together, these results prove the following theorem, which is the main result of this chapter.

Theorem 3.4. *Suppose M is a 3-connected matroid. If M has every element in a 3-cocircuit and every pair of elements in a 4-circuit, then M is one of the following matroids: $U_{3,5}$, $M(K_4)$, $\mathcal{W}^3$, F_7, $(F_7^-)^*$, P_7^*, and $M(K_3, n)$ for some $n \geq 3$.*

One property of matroids that we will exploit repeatedly is the restriction on circuit-cocircuit intersection, commonly referred to as orthogonality.

Theorem 3.5. *If $C \in \mathcal{C}(M)$ and $D \in \mathcal{C}(M^*)$, then $|C \cap D| \neq 1$.*

We shall need the following useful theorem of Lucas 3.6, which uses weak maps where, for two matroid M_1 and M_2 on the same set, the latter is a *weak-map image* of the former if every set that is independent in M_2 is also independent in M_1.

Theorem 3.6. *Let M_2 be the weak-map image of a binary matroid M_1 and suppose that $r(M_2) = r(M_1)$. Then M_2 is binary, and if $M_2 \neq M_1$, then M_2 is disconnected.*

3.2 Structure Lemmas

Our ability to determine M explicitly will rely heavily on being able to determine the arrangement of the 3-cocircuits of M. The mindful reader will note the approach taken here, as this chapter is something of a warm-up to Chapter 4. We first prove that should M have any 3-cocircuits that meet in two elements M must be $U_{3,5}$.

Proposition 3.7. *There exist 3-cocircuits D_1 and D_2 of M such that $|D_1 \cap D_2| = 2$ if and only if $M \cong U_{3,5}$. Moreover, if $|E(M)| \leq 5$, then $M \cong U_{3,5}$.*

Proof. If M is $U_{3,5}$, then it certainly has a pair of 3-cocircuits meeting in two elements. Now, suppose M has two cocircuits such D_1 and D_2. Let $E(M) = \{x_1, x_2, \ldots, x_n\}$. Without loss of generality, $D_1 = \{x_1, x_2, x_3\}$ and $D_2 = \{x_1, x_2, x_4\}$. Note that $D_1 \cup D_2$ is a 4-point line in M^*, and, therefore, any circuit meeting $D_1 \cup D_2$ must do so in at least three elements by orthogonality.

If $|E(M)| = 4$, then $M \cong U_{2,4}$, a contradiction since M must have a 4-circuit. Hence, we may assume $|E(M)| \geq 5$. By $(P2)$, we have a 4-circuit C_1 containing $\{x_1, x_5\}$. Without loss of generality, $C_1 = \{x_1, x_2, x_3, x_5\}$. Similarly, there is a 4-circuit C_2 containing $\{x_4, x_5\}$. Without loss of generality, we may assume $C_2 = \{x_1, x_2, x_4, x_5\}$. Then $r(C_1 \cup C_2) = 3$. Therefore $\lambda(C_1 \cup C_2) = r(C_1 \cup C_2) + r^*(C_1 \cup C_2) - |C_1 \cup C_2| \leq 3 + 2 - 4 = 1$. This implies $|E(M)| \leq 5$, since M is 3-connected. Thus M must be the 5-point plane $U_{3,5}$.

In order to see that $U_{3,5}$ is the only possibility when $|E(M)| = 5$, we only need to determine what happens when such an M has no two 3-cocircuits meeting in two elements. In this case, we get cocircuits $D_1 = \{x_1, x_2, x_3\}$ and $D_2 = \{x_1, x_4, x_5\}$ without loss of generality. Circuit elimination on this pair indicates there is a cocircuit contained in $\{x_2, x_3, x_4, x_5\}$. This cocircuit cannot have 3 elements without contradicting our assumption. Further, M cannot have a cocircuit of size 4 since $r*(M) \leq 2$. This contradiction completes the proof. $\square$

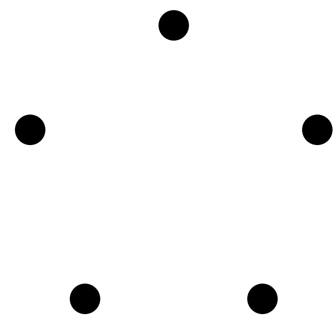


Figure 3.1: The matroid $U_{3,5}$

This result yields the following useful corollary concerning triangles in M.

Corollary 3.8. *If $|E(M)| \neq 6$, then M contains no triangles.*

Proof. Proposition 3.7 handles the case in which $|E(M)| \leq 5$.

Let $E(M) = \{x_1, x_2, \ldots, x_n\}$ for some $n \geq 7$, and suppose $T = \{x_1, x_2, x_3\}$ is a triangle in M. By $(P1)$ there is a 3-cocircuit D_1 containing x_1. By orthogonality, $|D_1 \cap T| > 1$, and by the 3-connectedness of M we know $T \neq D_1$. Therefore, we may assume that $D_1 = \{x_1, x_2, x_4\}$. Similarly, there is a 3-cocircuit D_2 containing x_3. Without loss of generality, $x_1 \in D_2$, and by Proposition 3.7 we may assume $D_2 = \{x_1, x_3, x_5\}$. Now, $(P1)$ guarantees a 4-circuit C containing $\{x_2, x_3\}$. As $T \nsubseteq C$, we must have $C = \{x_2, x_3, x_4, x_5\}$ by orthogonality. However, this means $\lambda(\{x_1, x_2, x_3, x_4, x_5\}) \leq 3 + 3 - 5 = 1$, a contradiction.

$\square$

The next proposition addresses the case when M has two disjoint 3-cocircuits. Specifically, we prove that two such 3-cocircuits must be locally isomorphic to $M(K_{2,3})$. This structure is the foundation for the main result, and will feature heavily in its proof.

Proposition 3.9. *If M has two disjoint 3-cocircuits D_1 and D_2, then $M|(D_1 \cup D_2) \cong M(K_{2,3})$.*

Proof. Let $E(M) = \{x_1, x_2, \ldots, x_n\}$, and suppose $D_1 = \{x_1, x_2, x_3\}$ and $D_2 = \{x_4, x_5, x_6\}$. As M is 3-connected, $n \geq 7$. By $(P1)$, there exists a 4-circuit C_1 containing $\{x_1, x_4\}$. By orthogonality, we may assume $C_1 = \{x_1, x_2, x_4, x_5\}$. Similarly, there is a 4-circuit C_2 containing $\{x_3, x_6\}$. By symmetry, we may assume $C_2 = \{x_1, x_3, x_4, x_6\}$. Lastly, there must be a 4-circuit C_3 containing $\{x_2, x_6\}$. Note that

Claim 3.9.1. *C_3 must not meet either C_1 or C_2 in three elements.*

Assume it does; that is, without loss of generality, $|C_1 \cap C_3| = 3$. Then $C_1 \cup C_3$ is a 5-point plane, in which there exists a 4-circuit meeting one of D_1 or D_2 in exactly one element. This contradiction proves the claim.

Therefore, neither x_1 nor x_4 can be in C_3, and we get $C_3 = \{x_2, x_3, x_5, x_6\}$. We will now apply Theorem 3.6 in order to complete the proof.

First, note that $r(M|(D_1 \cup D_2)) = 4$, as each 3-cocircuit is an independent hyperplane. Next, let $K_{2,3}$ be labeled as in Figure 3.2 and suppose $M(K_{2,3})$ inherits the edge labels. Evidently, the identity map $i : E(M(K_{2,3})) \to E(M)$ is a weak map, and, since M is 3-connected, it must be that $M \cong M(K_{2,3})$. □

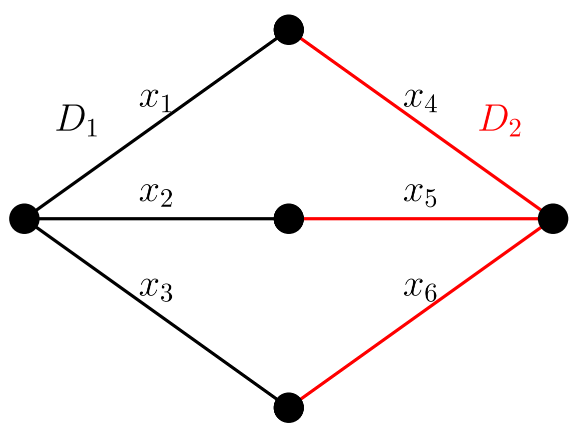


Figure 3.2: The graph $K_{2,3}$

Our final observation about the structure of M is that M must have three pairwise-disjoint 3-cocircuits when $|E(M)| \geq 9$. We build up to this in three steps. First, we prove a preliminary lemma and some subsequent corollaries revealing the necessary restrictions on the interaction between the 4-circuits and 3-cocircuits of M. The lemma indicates that there are no 3-cocircuits contained in 4-circuits. Following that, we prove that M is guaranteed to have at least two disjoint 3-cocircuits when $|E(M)| \geq 9$. We then extend this result to ensure three pairwise-disjoint 3-cocircuits, and, further, prove that they produce a local $M|M(K_{3,3})$-structure.

Lemma 3.10. *If $|E(M)| \geq 9$ and M has C as a 4-circuit and D as a 3-cocircuit, then $D \nsubseteq C$.*

Proof. Suppose not. Let $E(M) = \{x_1, x_2, \ldots, x_n\}$ for some $n \geq 9$, and suppose we have a 3-cocircuit $D_1 = \{x_1, x_2, x_3\}$ and a 4-circuit $C_1 = D_1 \cup \{x_4\}$. By $(P1)$, we are guaranteed a 3-cocircuit D_2 containing x_4. By orthogonality, D_2 must contain a second element of C_1, and by Proposition 3.7 it has at most one element in common with D_1. Hence, we may assume $D_2 = \{x_1, x_4, x_5\}$.

Now, there is a 4-circuit C_2 containing $\{x_2, x_5\}$. By orthogonality, either $x_1 \in C_2$, or $C_2 = \{x_2, x_3, x_4, x_5\}$. Note, however, that a second 4-circuit contained in $\{x_1, x_2, x_3, x_4, x_5\}$ means $\lambda(\{x_1, x_2, x_3, x_4, x_5\}) = r(\{x_1, x_2, x_3, x_4, x_5\}) + r^*(\{x_1, x_2, x_3, x_4, x_5\}) - |\{x_1, x_2, x_3, x_4, x_5\}| \leq 3 + 3 - 5 = 1$. This implies that $x_1 \in C_2$, and, further, that $C_2 = \{x_1, x_2, x_5, x_6\}$.

Next, consider a 4-circuit C_3 containing $\{x_3, x_5\}$. By orthogonality, $x_1 \in C_3$, and by the above argument we know $C_3 \nsubseteq \{x_1, x_2, x_3, x_4, x_5\}$. Therefore, either $C_3 = \{x_1, x_3, x_5, x_6\}$, or $C_3 = \{x_1, x_3, x_5, x_7\}$. We show next that

Claim 3.10.1. $C_3 = \{x_1, x_3, x_5, x_7\}$.

Suppose not, that $C_3 = \{x_1, x_3, x_5, x_6\}$. Then $\{x_1, x_2, x_3, x_5, x_6\}$ is a 5-point plane, implying $\{x_1, x_2, x_5, x_6\}$ is a circuit. This violates orthogonality with D_2, a contradiction.

Finally, consider a 3-cocircuit D_3 containing x_6. By orthogonality with C_2, we have $\{x_1, x_2, x_5\} \cap D_3 \neq \emptyset$. We will show that the inclusion of any of x_1, x_2, or x_5 in D_3 produces a contradiction. If $x_1 \in D_3$, then this forces $D_3 = \{x_1, x_3, x_6\}$ by orthogonality with C_1 and C_3, and this contradicts Proposition 3.7. Therefore, $x_1 \notin D_3$. If $x_2 \in D_3$, then $D_3 = \{x_2, x_4, x_6\}$ by orthogonality with C_1. However, now $\lambda(\{x_1, x_2, \ldots, x_6\}) \leq 4 + 3 - 6 = 1$, a contradiction. Therefore, $x_2 \notin D_3$. Lastly, suppose $x_5 \in D_3$. By orthogonality with C_3, one of x_3 and x_7 must be in D_3. The inclusion of x_3 contradicts orthogonlity with C_1, so $D_3 = \{x_5, x_6, x_7\}$. But now, $\lambda(\{x_1, x_2, \ldots, x_7\}) \leq 4 + 4 - 7 = 1$, a contradiction. Thus there is no 3-cocircuit containing x_6, and this contradiction proves the lemma.

□

Corollary 3.11. *If $|E(M)| \geq 9$, then M contains no 5-point planes.*

Proof. Let S be a 5-point plane in M and suppose $e \in S$. Then by $(P1)$ there must be a 3-cocircuit containing e, and, by orthogonality, that cocircuit must be contained in S, a contradiction to the last lemma. □

Corollary 3.12. *If $|E(M)| \geq 9$, then M contains no 4-point colines.*

Proof. Let S be a 4-point coline in M and suppose $\{x_1, x_2\} \subseteq S$. Then by $(P1)$ there must be a 4-circuit containing $\{x_1, x_2\}$, and, by orthogonality, that circuit must contain at least one additional element of S, a contradiction. □

Lemma 3.13. *If $|E(M)| \geq 9$, then M has two disjoint 3-cocircuits.*

Proof. Suppose not. Let $E(M) = \{x_1, x_2, \ldots, x_n\}$. Let D_1 and D_2 be distinct 3-cocircuits of M. By Proposition 3.7, $|D_1 \cap D_2| \leq 1$, so we may assume $D_1 = \{x_1, x_2, x_3\}$ and $D_2 = \{x_1, x_4, x_5\}$. We demonstrate that

Claim 3.13.1. *there is a third 3-cocircuit containing x_1.*

Suppose not. By $(P1)$ there is a 3-cocircuit D_3 containing x_6. By assumption, D_3 meets each of D_1 and D_2. Then, $D_3 = \{x_2, x_4, x_6\}$, without loss of generality. But $(P1)$ further guarantees a 3-cocircuit containing x_7. By the pigeonhole principle, such a cocircuit cannot meet each of D_1, D_2, and D_3 without using an element shared by two of them, thus proving the claim.

Therefore, we may assume that $D_3 = \{x_1, x_6, x_7\}$. Now, consider a 3-cocircuit D_4 containing x_8. In order to meet each of D_1, D_2, and D_3, it must be that $x_1 \in D_4$, and so we may assume $D_4 = \{x_1, x_8, x_9\}$. However, $(P1)$ guarantees a 4-circuit C containing $\{x_1, x_2\}$. By orthogonality, C must contain a second element from each of D_2, D_3, and D_4, implying $|C| = 5$. This contradiction proves the lemma. $\square$

Proposition 3.14. *If $|E(M)| \geq 9$, then M contains three pairwise-disjoint 3-cocircuits.*

Proof. Suppose not. Let $E(M) = \{x_1, x_2, \ldots, x_n\}$. By Lemma 3.13, we get disjoint 3-cocircuits D_1 and D_2. We may assume $D_1 = \{x_1, x_2, x_3\}$ and $D_2 = \{x_4, x_5, x_6\}$. By Proposition 3.9, $M|(D_1 \cup D_2) \cong M(K_{2,3})$. Therefore we get circuits $C_1 = \{x_1, x_2, x_4, x_5\}$, $C_2 = \{x_1, x_3, x_4, x_6\}$, and $C_3 = \{x_2, x_3, x_5, x_6\}$, without loss of generality. Any additional 3-cocircuit must meet $D_1 \cup D_2$, and, by orthogonality, it must do so in one of the series pairs of $M|(D_1 \cup D_2)$. Therefore, if D_3 is a 3-cocircuit containing x_7, we may assume $D_3 = \{x_1, x_4, x_7\}$. Similarly, if D_4 is a 3-cocircuit containing x_8, we may assume $D_4 = \{x_2, x_5, x_8\}$, since D_4 must not contain $\{x_1, x_4\}$ by Proposition 3.7. Finally, if D_5 is a 3-cocircuit containing x_9, it must be that $D_5 = \{x_3, x_6, x_9\}$. But then D_3, D_4, and D_5 are disjoint, a contradiction.

$\square$

Lemma 3.15. *Let $D_1 = \{x_1, x_2, x_3\}$, $D_2 = \{x_4, x_5, x_6\}$, and $D_3 = \{x_7, x_8, x_9\}$ be cocircuits of M. Suppose the sets of 4-circuits contained in $M|(D_1 \cup D_2)$ and $M|(D_1 \cup D_3)$ are $\{\{x_1, x_2, x_4, x_5\}, \{x_1, x_3, x_4, x_6\}, \{x_2, x_3, x_5, x_6\}\}$ and $\{\{x_1, x_2, x_7, x_8\}, \{x_1, x_3, x_7, x_9\}, \{x_2, x_3, x_8, x_9\}\}$. Then the 4-circuits contained in $M|(D_2 \cup D_3)$ are $\{\{x_4, x_5, x_7, x_8\}, \{x_4, x_6, x_7, x_9\}, \{x_5, x_6, x_8, x_9\}\}$, and $M|(D_1 \cup D_2 \cup D_3) = M(K_{3,3})$, where the vertex bonds of $K_{3,3}$ are D_1, D_2, D_3, $\{x_1, x_4, x_7\}$, $\{x_2, x_5, x_8\}$, and $\{x_3, x_6, x_9\}$.*

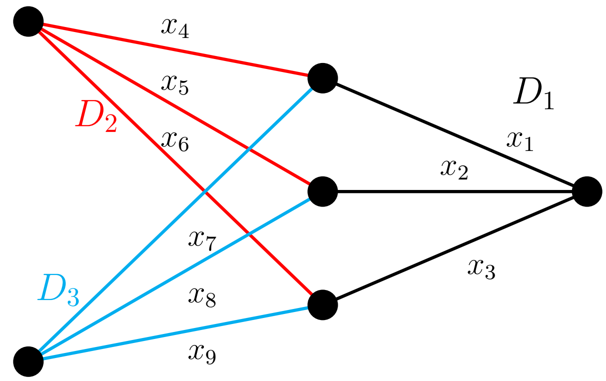


Figure 3.3: The graph $K_{3,3}$ provides the underlying structure to three disjoint 3-cocircuits

Proof. In order to prove $M|(D_1 \cup D_2 \cup D_3) \cong M(K_{3,3})$, we will recruit Theorem 3.6. To that end, we must determine the circuits in $M|(D_1 \cup D_2 \cup D_3)$. Specifically, we will find the nine 4-circuits and six 6-circuits that exist in $M(K_{3,3})$. See Figure 3.3 for reference.

By Lemma 3.9, we know that M restricted to each pair of disjoint 3-cocircuits is isomorphic to $M(K_{2,3})$. Hence, we may assume $C_1 = \{x_1, x_2, x_4, x_5\}$, $C_2 = \{x_1, x_3, x_4, x_6\}$, and $C_3 = \{x_2, x_3, x_5, x_6\}$ are the circuits in $M|(D_1 \cup D_2)$. Further, we may assume that $C_4 = \{x_1, x_2, x_7, x_8\}$, $C_5 = \{x_1, x_3, x_7, x_9\}$, and $C_6 = \{x_2, x_3, x_8, x_9\}$ are the 4-circuits in $M|(D_1 \cup D_3)$, without loss of generality. The 4-circuits in $M|(D_2 \cup D_3)$ can then be determined by circuit elimination. First, take the circuit $C_7 \subseteq (C_1 \cup C_4) - x_1$. By othogonality, $C_7 = \{x_4, x_5, x_7, x_8\}$. Similarly, the circuit $C_8 \subseteq (C_2 \cup C_5) - x_1$ must be $C_8 = \{x_4, x_6, x_7, x_9\}$. Lastly, the circuit $C_9 \subseteq (C_3 \cup C_6) - x_2$ must be $C_9 = \{x_5, x_6, x_8, x_9\}$. The six 6-circuits are, by orthogonlaity, precisely the following sets: $(C_1 \cup C_8) - x_4$, $(C_1 \cup C_9) - x_5$, $(C_2 \cup C_7) - x_4$, $(C_2 \cup C_9) - x_6$, $(C_3 \cup C_7) - x_5$, and $(C_3 \cup C_8) - x_6$.

Now, let $K_{3,3}$ be labeled as in Figure 3.3, and have $M(K_{3,3})$ inherit the edge labels. Then, the identity map $i : E(M(K_{3,3})) \to E(M)$ is a weak map, and, since $r(M) = r(M \backslash D_1) + 1 = r(M(K_{2,3})) + 1 = r(M(K_{3,3}))$, Theorem 3.6 indicates $M \cong M(K_{3,3})$. $\square$

3.3 When M Has Few Elements

Here will shall determine all matroids with property $(P1)$ having fewer than nine elements. Part of the job is done, by Proposition 3.7. We first show that, if M has disjoint 3-cocircuits, then it must be the matroid P_7^*. Afterwards, we will find a couple of matroids on six elements, and a couple of matroids on seven elements. There are no matroids on eight elements that have property $(P1)$.

Proposition 3.16. *If $|E(M)| \leq 8$ and M has two disjoint 3-cocircuits, then $M \cong P_7^*$.*

Proof. The result is immediate if $|E(M)| \leq 5$.

Let $E(M) = \{x_1, x_2, \ldots, x_n\}$, and suppose that M has a pair of disjoint 3-cocircuits D_1 and D_2. Let $M|(D_1 \cup D_2)$ be labeled as in Figure 3.2. We take the proof in three cases, first ruling out the 6- and 8-element cases. If $|E(M)| = 6$, then by Lemma 3.9, $M \cong M(K_{2,3})$. This matroid is not 3-connected, contradicting $(P1)$.

Assume, next, that $|E(M)| = 8$. By $(P1)$, there is a 3-cocircuit D_3 containing x_7. As M has only eight elements, D_3 meets $D_1 \cup D_2$ and, by orthogonality, must do so in a series pair of $M|(D_1 \cup D_2)$. Without loss of generality, $D_3 = \{x_1, x_4, x_7\}$. Similarly, a 3-cocircuit D_4 containing x_8 must meet $D_1 \cup D_2$ in a series pair, and that pair cannot be $\{x_1, x_4\}$ by Proposition 3.7. Now, there is a 4-circuit C containing $\{x_3, x_7\}$. By orthonality, either $x_1 \in C$, or $C = \{x_2, x_3, x_4, x_7\}$. The latter case is out by orthogonality with D_4, so suppose $x_1 \in C$. Each of the remaining elements in $E(M) - \{x_1, x_3, x_7\}$ are in some cocircuit disjoint from $\{x_1, x_3, x_7\}$. Thus, C cannot include any additional elements without violating orthogonality.

Now, we may assume $|E(M)| = 7$. We begin as in the previous case. By $(P2)$, x_7 is in a 3-cocircuit D_3. Then, by orthogonality, D_3 meets $D_1 \cup D_2$ in a series pair of $M|(D_1 \cup D_2)$. Without loss of generality, $D_3 = \{x_1, x_4, x_7\}$. There must be a circuit C_1 $\{x_1, x_7\}$. By orthogonality, this is forced to be $C_1 = \{x_1, x_2, x_3, x_7\}$. Hence, $x_7 \in \mathrm{cl}(D_1)$, and it follows that $r(M) = 4$. Therefore, the complement of each 4-circuit of M is a 3-cocircuit, and the

3-cocircuits align as in Figure 3.4. This structure admits no further 3-cocircuits, and since $r^*(M) = 3$, we have determined the full list of cocircuits of M. Thus $M \cong P_7^*$, and the lemma is proved. $\square$

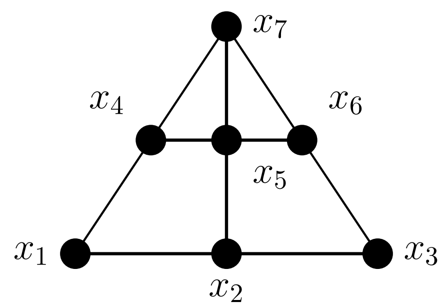


Figure 3.4: The matroid P_7.

For the remainder of the section we need only consider matroids having no two disjoint 3-cocircuits. With this last proposition, we will have determined all matroids having property $(P1)$ on fewer than nine elements.

Proposition 3.17. *If* $6 \leq |E(M)| \leq 8$ *and* M *has no two disjoint 3-cocircuits, then* M *is one of the following matroids:* $M(K_4)$, $\mathcal{W}^3$, F_7^*, *and* $(F_7^-)^*$.

Proof. Let $E(M) = \{x_1, x_2, \ldots, x_n\}$ for some $n \in \{6, 7, 8\}$. Without loss of generality, we may assume $D_1 = \{x_1, x_2, x_3\}$ and $D_2 = \{x_1, x_4, x_5\}$ are cocircuits. We proceed in cases by the size of M.

If $|E(M)| = 6$, then, without loss of generality, we may assume a cocircuit D_3 containing x_6 is $\{x_2, x_4, x_6\}$. Now $r(M) = r^*(M) = 3$, so any the complement of any 3-cocircuit of M is a triangle. Should D_1, D_2, and D_3 be the complete list of 3-cocircuits of M, then, evidently, $M \cong \mathcal{W}^3$. However, this structure also admits $\{x_3, x_5, x_6\}$ as a cocircuit without violating $(P1)$. In this case, $M \cong M(K_4)$.

Suppose $|E(M)| = 7$. In this case, either $r(M) = 3$, or $r(M) = 4$. In the former case, hyperplanes of M will have rank 2. Therefore, the complement of each 3-cocircuit is a 4-

point line; but, if both $\{x_4, x_5, x_6, x_7\}$ and $\{x_2, x_3, x_6, x_7\}$ are 4-point lines, then their union is a 6-point line, a contradiction. Hence $r(M) = 3$.

Now, all 4-circuits of M are hyperplanes. We get circuits $C_1 = E(M) - D_1$ and $C_2 = E(M) - D_2$ immediately. By $(P1)$ we are guaranteed a 4-circuit C_3 containing $\{x_1, x_5\}$. By orthogonality, it suffices to assume $C_3 = \{x_1, x_3, x_5, x_7\}$. Hence, $D_3 = \{x_2, x_4, x_6\}$ is a cocircuit. Similarly, a 4-circuit C_4 containing $\{x_2, x_5\}$ can be assumed to be $C_4 = \{x_2, x_4, x_5, x_6\}$, without loss of generality. This yields the cocircuit $D_4 = \{x_1, x_2, x_7\}$. Further, a 4-circuit C_5 containing $\{x_3, x_7\}$ is either $\{x_3, x_4, x_6, x_7\}$ or $\{x_2, x_3, x_5, x_7\}$. These are symmetric, so we may assume $C_5 = \{x_3, x_4, x_6, x_7\}$ and that $D_5 = \{x_1, x_2, x_5\}$ is a cocircuit. The only pair of elements not yet in some 4-circuit is $\{x_5, x_7\}$. Suppose C_6 is the 4-circuit containing them. Again, this circuit must be either $\{x_1, x_4, x_5, x_7\}$ or $\{x_2, x_3, x_5, x_7\}$, and these sets are symmetric. Thus, we may let $C_6 = \{x_1, x_4, x_5, x_7\}$. Such a matroid M having $\mathcal{C}(M) = \{C_1, C_2, C_3, C_4, C_5, C_6\}$ is isomorphic to $(F_7^-)*$. It is possible, however, that M admits a seventh 4-circuit, $C_7 = \{x_2, x_3, x_5, x_7\}$. This produces the matroid F_7^*, and concludes this case.

Lastly, we assume $|E(M)| = 8$. Here we will first prove that M has three 3-cocircuits meeting in a shared element. If not, then, without loss of generality, a 3-cocircuit D_3 containing x_6 is $D_3 = \{x_2, x_4, x_6\}$. In order to meet each of the previous 3-cocircuits but not use any element already shared between them, a cocircuit D_4 containing x_7 must be $\{x_3, x_5, x_7\}$. But $(P1)$ guarantees a 3-cocircuit containing x_8, and all other elements of M are already in two 3-cocircuits.

Therefore, we may assume that a 3-cocircuit D_3 containing x_6 is $D_3 = \{x_1, x_6, x_7\}$. But now, a 3-cocircuit D_4 containing x_8 again leads to contradiction, as, in order to meet all other 3-cocircuits, it must be that $x_1 \in D_4$. Then, D_4 is forced to meet one of D_1, D_2, and D_3 in two elements, contradicting Proposition 3.7. Thus there are no matroids on eight elements, and the proof is complete.

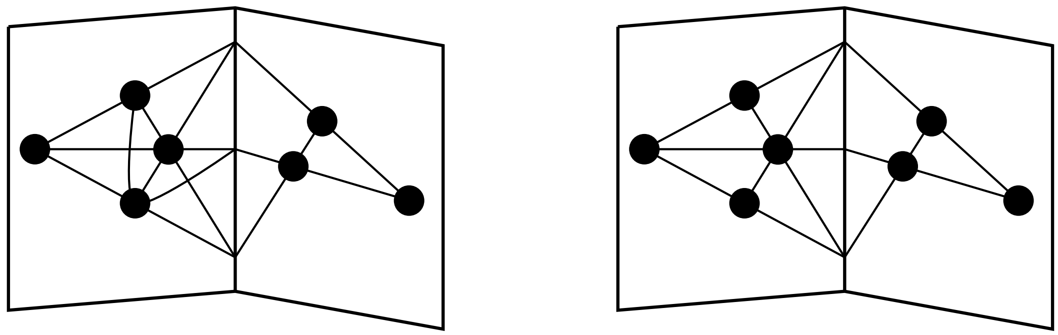


Figure 3.5: The matroids F_7^* and $(F_7^-)^*$

$\square$

3.4 The Main Result

All that remains is to determine M when $|E(M)| \geq 10$. We first prove that such an M can be partitioned into 3-cocircuits. Using these disjoint cocircuits, we will be able to complete an induction argument to prove the final component of the main result.

Lemma 3.18. *If $|E(M)| \geq 9$, then $E(M)$ can be partitioned into $D_1 \cup D_2 \cup \cdots \cup D_n$, where each D_i is a 3-cocircuit.*

Proof. If $|E(M)| = 9$, then by Proposition 3.14 we are done.

Now suppose $|E(M)| > 9$, and let $S = \{D_1, D_2, \ldots, D_n\}$ be the largest collection of pairwise-disjoint 3-cocircuits of M. Let e be an element not in any D_i, for $i \in \{1, 2, \ldots, n\}$. By $(P1)$, there is a 3-cocircuit D_e containing e. As D_e must meet some cocircuit in S, we may assume that $D_e \cap D_1 \neq \emptyset$. By Lemma 3.9, $M|(D_1 \cup D_i) \cong M(K_{2,3})$ for all $i \in \{2, 3, \ldots, n\}$. By orthogonality, D_e must meet each of these local $M(K_{2,3})$'s in a series pair, a contradiction. Thus, the lemma is proved. $\square$

Proposition 3.19. *If $|E(M)| \geq 9$, then $M \cong M(K_{3,n})$ for some $n \geq 3$.*

Proof. By Lemma 3.18, for some $n \geq 3$, there is a partition of $E(M)$ into 3-cocircuits $D_1, D_2, \ldots, D_n$ where $D_i = \{x_i, y_i, z_i\}$ for all i. By Proposition 3.9, as $M|(D_1 \cup D_2) \cong M(K_{2,3})$

when $i \neq j$, we can assume that M has $\{x_1, x_2, y_1, y_2\}$, $\{x_1, x_2, z_1, z_2\}$, and $\{y_1, y_2, z_1, z_2\}$ as circuits. By repeatedly applying Lemma 3.15, we can assume that M has $\{x_i, x_j, y_i, y_j\}$, $\{x_i, x_j, z_i, z_j\}$, and $\{y_i, y_j, z_i, z_j\}$ as circuits for all $1 \leq i < j < n$. We prove by induction on k that, for $3 \leq k \leq n$, with $K_{3,k}$ labelled so tht its vertex bonds are $D_1, D_2, \dots, D_k$, $\{x_1, x_2, \dots, x_k\}$, $\{y_1, y_2, \dots, y_k\}$, and $\{z_1, z_2, \dots, z_k\}$, we have $M|(D_1 \cup D_2 \cup \dots \cup D_n) = M(K_{3,k})$.

By Lemma 3.15, this is true when $k = 3$. Assume it is true for $k < m$, and let $k = m \geq 3$. Suppose $M|(D_1 \cup D_2 \cup \dots \cup D_m) \neq M(K_{3,m})$. Let Z be a minimal set that is independent in one of $M|(D_1 \cup D_2 \cup \dots \cup D_m)$ and $M(K_{3,m})$ and dependent in the other. Then Z is independent in one matroid, say M_I, and a circuit in the other, say M_C.

As $M|(D_1 \cup D_2 \cup \dots \cup D_{m-1}) = M(K_{3,m-1})$, it follows by the induction assumption that Z meets D_m. By symmetry, Z meets each of $D_1, D_2, \dots, D_{m-1}$. As $D_1, D_2, \dots, D_m$ are cocircuits of each $M|(D_1 \cup D_2 \cup \dots \cup D_m)$ and $M(K_{3,m})$, it follows, by orthogonality in M_C, that Z meets each of $D_1, D_2, \dots, D_{m-1}$ in at least two elements. Therefore $|Z| \geq 2m$. But

$$\begin{aligned} r(M|(D_1 \cup D_2 \cup \dots \cup D_m)) &= r(M(K_{3,m-1})) + 1 \\ &= m + 2. \end{aligned}$$

Hence $|Z| \leq m + 3$. Thus $2m \leq |Z| \leq m + 2$, so $m \leq 2$; a contradiction. The result follows by induction.

□

Upon combining the previous propositions, we get the main result.

Theorem 3.20. *Suppose M is a non-empty 3-connected matroid. If M has every element in a 3-cocircuit and every pair of elements in a 4-circuit, then M is one of the following matroids: $U_{3,5}$, $M(K_4)$, $\mathcal{W}^3$, F_7, $(F_7^-)^*$, P_7^*, and $M(K_{3,n})$ for some $n \geq 3$.*

Chapter 4
Many 4-cocircuits and 4-circuits

4.1 Introduction and Preliminaries

This chapter continues the line of inquiry from Chapter 3. We turn our focus now to matroids M having the following property:

4.1. A matroid M has property $(P2)$ if, for all distinct elements e and f of M:

(i) there exists some 4-circuit $C \in \mathcal{C}(M)$ such that $\{e, f\} \subseteq C$,

(ii) there exists some 4-cocircuit $D \in \mathcal{C}(M^*)$ such that $e \in D$, and

(iii) M is 4-connected.

We will assume throughout this chapter that M has property $(P2)$, and will proceed to determine all such matroids. The necessary preliminaries are the same as in Chapter 3, and we will take a similar approach with the arguments. As before, we will begin by making several observations about the structural consequences of $(P2)$, and then proceed to determine explicitly the possibilities for M when $|E(M)|$ is small. More precisely, we show that M is either $U_{3,6}$, one of 21 paving matroids with eight elements, one of 10 matroids with nine elements, the matroid R_10, a matroid on twelve elements, a matroid on fourteen elements, or M has at least sixteen elements and $M \cong M(K_{4,n})$.

4.2 Structure Lemmas

The first proposition of this section addresses when M has two 4-cocircuits that meet in three elements, and shows that this structure only occurs in one case.

Proposition 4.2. *There exist 4-cocircuits D_1 and D_2 of M such that $|D_1 \cap D_2| = 3$ if and only if $M \cong U_{3,6}$.*

Proof. If $M \cong U_{3,6}$, it certainly has such a pair of 4-cocircuits. Now, suppose M contains two such 4-cocircuits. Let $E(M) = \{x_1, x_2, \ldots, x_n\}$. Without loss of generality, $D_1 = \{x_1, x_2, x_3, x_4\}$ and $D_2 = \{x_1, x_2, x_3, x_5\}$. Note that $D_1 \cup D_2$ is a 5-point plane in M^*, and, therefore, any circuit meeting $D_1 \cup D_2$ must do so in at least 3 elements, otherwise it will violate orthogonality.

We will first show that $D_1 \cup D_2$ must contain a circuit. Consider a 4-circuit, C_1, containing $\{x_1, x_2\}$. If C_1 is not contained in $D_1 \cup D_2$, then $C_1 = \{x_1, x_2, x_3, x_6\}$, without loss of generality. There must also be a 4-circuit, C_2, containing $\{x_4, x_6\}$. Clearly, $C_2 \subsetneqq D_1 \cup D_2 \cup \{x_6\}$. Then, by circuit elimination, there is a circuit $C_3 \subseteq (C_1 \cup C_2) - \{x_6\} \subseteq D_1 \cup D_2$, as desired.

Let Z be a 4-circuit that is contained in $D_1 \cup D_2$. Then Z is also a cocircuit, so $r(Z) + r^*(Z) - |Z| = 3 + 3 - 4 = 2$. Hence $|E(M) - Z| = 2$, so $|E(M)| = 6$. As M has no circuits of size less than three, it follows that $M \cong U_{3,6}$. $\square$

The next proposition states that having two disjoint 4-cocircuits in M ensures a local $K_{2,4}$-structure. We later use this local structure as a basis for the induction argument proving that, when M has a sufficient number of elements, it must be isomorphic to $M(K_{4,n})$ for some natural number $n \geq 4$.

The proof of this requires three preliminary lemmas, which each rule out a particular configuration of 4-circuits that might occur between the disjoint 4-cocircuits. In each of these lemmas, $X = \{x_1, x_2, x_3, x_4\}$ and $Y = \{y_1, y_2, y_3, y_4\}$ are disjoint 4-cocircuits of M. Observe that orthogonality and the 4-connectedness of M implies that every 4-circuit contained in $X \cup Y$ meets each of X and Y in exactly two elements.

Lemma 4.3. *If C_1 and C_2 are distinct 4-circuits contained in $X \cup Y$ such that $|C_1 \cap C_2 \cap X| \geq 1$, then $|C_1 \cap C_2 \cap X| = 1$.*

Proof. Clearly $|C_1 \cap C_2 \cap X| \neq 4$. If $|C_1 \cap C_2 \cap X| = 3$, then each of C_1 and C_2 has only one element from Y, which contradicts their orthogonality with Y. Therefore $|C_1 \cap C_2 \cap X| \neq 3$.

Now suppose $|C_1 \cap C_2 \cap X| = 2$. Therefore, either $|C_1 \cap C_2| = 3$, or $|C_1 \cap C_2| = 2$. If $|C_1 \cap C_2| = 3$, then, without loss of generality, $C_1 = \{x_1, x_2, y_1, y_2\}$, and $C_2 = \{x_1, x_2, y_1, y_3\}$. By circuit elimination, there is a circuit contained in $\{x_1, y_1, y_2, y_3\}$, and such a circuit will violate either the 4-connectedness of M, or orthogonality. If $|C_1 \cap C_2| = 2$, then, without loss of generality, $C_1 = \{x_1, x_2, y_1, y_2\}$, and $C_2 = \{x_1, x_2, y_3, y_4\}$. This implies, by circuit elimination, that there is a circuit contained in $\{x_1, y_1, y_2, y_3, y_4\}$. Such a circuit must contain x_1, and thus will contradict orthogonality with the cocircuit X. □

The last lemma shows that pairs of elements from X or from Y will occur at most once among the 4-circuits contained within $X \cup Y$.

We must now determine how those pairs of elements from each 4-cocircuit match up within the 4-circuits that contain them. We achieve this in the following two results. The next lemma indicates that three circuits contained in $X \cup Y$ cannot cover X unless they also cover Y.

Lemma 4.4. *If C_1, C_2, and C_3 are distinct 4-circuits of M such that $C_1 \cup C_2 \cup C_3 \subseteq X \cup Y$ and $C_1 \cup C_2 \cup C_3 \supseteq X$, then $C_1 \cup C_2 \cup C_3 \supseteq Y$.*

Proof. Suppose not. Then, by Lemma 4.3, we may assume that $C_1 \cap Y = \{y_2, y_3\}$, $C_2 \cap Y = \{y_1, y_3\}$, and $C_3 \cap Y = \{y_1, y_2\}$. Without loss of generality, we may assume that $C_1 \cap X = \{x_1, x_2\}$ and $C_2 \cap X = \{x_1, x_3\}$. Then $\{x_1, y_1, y_2, y_3\}$ spans X. As X is independent by the 4-connectedness of M, we have that X spans $X \cup Y$. This implies that, for any $y \in Y$,there is a circuit containing y and contained in $X \cup \{y\}$. This contradicts orthogonality, and thus the lemma is proved. □

Lemma 4.5. *If C_1 and C_2 are distinct 4-circuits in $X \cup Y$ such that $|C_1 \cap C_2| \geq 1$, then $|C_1 \cap C_2| = 2$.*

Proof. Assume the lemma fails. As $|C_1 \cap C_2| = |C_1 \cap C_2 \cap X| + |C_1 \cap C_2 \cap Y|$, it follows by Lemma 4.3 and symmetry that we may assume $|C_1 \cap C_2 \cap X| = 1$ and $|C_1 \cap C_2 \cap Y| = 0$. Without loss of generality, $C_1 = \{x_1, x_2, y_1, y_2\}$ and $C_2 = \{x_1, x_3, y_3, y_4\}$. By Lemma 4.4, any further 4-circuit contained in $X \cup Y$ must contain x_4. Property $(P2)$ guarantees the existence of a 4-circuit C_3 containing $\{x_2, y_3\}$. By Lemma 4.3, we have $C_3 = \{x_2, x_4, y_1, y_3\}$, without loss of generality. Now $(P2)$ similarly guarantees a 4-circuit C_4 containing $\{x_2, y_4\}$. As noted above, it must be that $x_4 \in C_4$, but this contradicts Lemma 4.3. Thus, no such C_4 exists, and our assertion holds. □

The three previous lemmas combined yield the following result.

Proposition 4.6. *Let M be a matroid with property $(P2)$. If M has X and Y as disjoint 4-cocircuits, then $M|(X \cup Y) \cong M(K_{2,4})$.*

Proof. There is a 4-circuit, C_1, in M containing x_1 and y_1. Without loss of generality, $C_1 = \{x_1, x_2, y_1, y_2\}$. Similarly, there is a 4-circuit, C_2, in M containing x_1 and y_3. Without loss of generality, $C_2 = \{x_1, x_3, y_1, y_3\}$, since, by Lemma 4.5, C_1 and C_2 intersect in exactly two elements. There is a 4-circuit, C_3 containing x_1 and y_4. By Lemmas 4.3 and 4.5, $C_3 = \{x_1, x_4, y_1, y_4\}$.

A fourth 4-circuit, C_4, may be found containing x_2 and y_3. Since C_4 meets C_1 in x_2, we have $x_1 \notin C_4$ by Lemma 4.3. Therefore by Lemma 4.5, since C_4 meets C_2 in y_3, we have $y_1 \notin C_4$, so $x_3 \in C_4$, and, similarly, since C_4 meets C_1 in x_2, we have $y_2 \in C_4$. Hence $C_4 = \{x_2, x_3, y_2, y_3\}$. Similarly, the 4-circuit containing x_2 and y_4 must be $C_5 = \{x_2, x_4, y_2, y_4\}$, and the final 4-circuit, one containing x_3 and y_4, must be $C_6 = \{x_3, x_4, y_3, y_4\}$.

To see that $\mathcal{C}(M|(X \cup Y)) = \{C_1, C_2, \ldots, C_6\}$, first observe that, since every 2-element subset of each of X and Y is in one of $C_1, C_2, \ldots, C_6$, Lemma 4.3 implies that $M|(X \cup Y)$ has no other 4-circuits. Clearly $r(X \cup Y) = 5$. Suppose $C \in \mathcal{C}(M|(X \cup Y)) - \{C_1, C_2, \ldots, C_6\}$. If $|C| = 6$, then C contains some C_i; a contradiction. Therefore, $|C| = 5$. To maintain orthogonality, C must be comprised of two elements from one of X and Y, and three elements from the other. To avoid containing one of the six 4-circuits, we may assume that $C = \{x_1, x_2, y_2, y_3, y_4\}$. Then, $\mathrm{cl}(\{x_1, y_2, y_3, y_4\}) = \{x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4\}$, so $r(X \cup Y) = 4$, a contradiction.

With the structure of the circuits in $\mathcal{C}(M|(X \cup Y))$ determined, we are now able to show that $M|(X \cup Y) \cong M(K_{2,4})$. First note that $r(M|(X \cup Y)) = 5$. Then, with $K_{2,4}$ labeled as in Figure 4.1 and $M(K_{2,4})$ inheriting the edge labels, the map $\phi : E(M|(X \cup Y)) \to E(M(K_{2,4}))$, given by $\phi(x_i) = a_i$ and $\phi(y_i) = b_i$, is an isomorphism. Thus, by Theorem 3.6, we have $M|(X \cup Y) \cong M(K_{2,4})$, and the proposition holds. $\square$

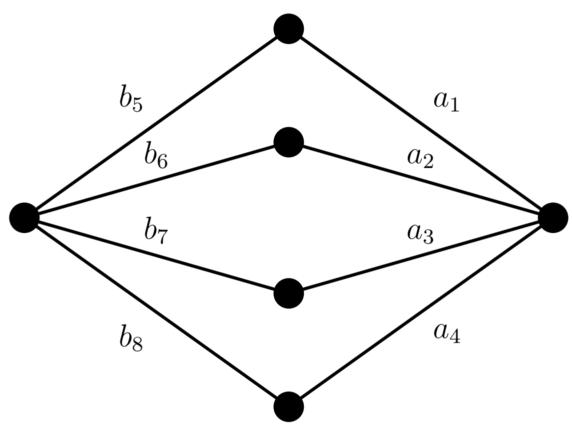


Figure 4.1: The graph $K_{2,4}$.

The third proposition of this section proves that the desirable $K_{2,4}$-structure must be present when $|E(M)| \geq 11$. The proof of this requires five preliminary results which restrict the ways in which 4-cocircuits may intersect. The first of these shows that three 4-cocircuits cannot pairwise meet in a common element unless those 4-cocircuits cover the matroid.

Lemma 4.7. *If $D_1, D_2,$ and D_3 are 4-cocircuits of M such that $|D_1 \cap D_2 \cap D_3| = 1$ and $|D_i \cap D_j| = 1$ for $i, j \in \{1, 2, 3\}$ with $i \neq j$, then $E(M) = D_1 \cup D_2 \cup D_3$.*

Proof. Suppose D_1, D_2, and D_3 as above, and yet $E(M) - (D_1 \cup D_2 \cup D_3) \neq \emptyset$. Take $e \in E(M) - (D_1 \cup D_2 \cup D_3)$, and $x \in D_1 \cap D_2 \cap D_3$. There is a 4-circuit, C, containing $\{e, x\}$, and C must contain at least two elements from each D_i by orthogonality. This forces C to have at least five elements, a contradiction. □

Building on the previous lemma, we demonstrate that M must have two 4-cocircuits that meet in two elements when $|E(M)| \geq 11$ and M has no disjoint 4-cocircuits. Then, we show that the union of two such 4-cocircuits meets every other 4-cocircuit in at least two elements.

Lemma 4.8. *If M has no two disjoint 4-cocircuits and $|E(M)| \geq 11$, then there exist 4-cocircuits D_1 and D_2 of M such that $|D_1 \cap D_2| = 2$.*

Proof. Suppose not. Let $E(M) = \{x_1, x_2, \ldots, x_n\}$, where $n \geq 11$. The element x_1 is in a 4-cocircuit; without loss of generality, that cocircuit is $D_1 = \{x_1, x_2, x_3, x_4\}$. We also have a 4-cocircuit, D_2, that contains x_5 and meets D_1. By Proposition 4.2, $|D_2 \cap D_1| = 1$. Without loss of generality, $D_2 = \{x_1, x_5, x_6, x_7\}$. Similarly, there is a 4-cocircuit, D_3, that contains x_8 and meets both D_1 and D_2 in a single element. By Lemma 4.7, we know $x_1 \notin D_3$. Therefore, without loss of generality, $D_3 = \{x_2, x_5, x_8, x_9\}$. Lastly, there is a 4-cocircuit, D_4, containing x_{10} and meeting each of D_1, D_2, and D_3 in a single element. Lemma 4.7 forces $D_4 = \{x_3, x_6, x_8, x_{10}\}$, without loss of generality. Then, the 4-cocircuit containing x_{11} must contain $\{x_4, x_7, x_9, x_{11}\}$. Thus D_4 and D_5 are disjoint, a contradiction. □

Note that the next three lemmas only require M to have at least 10 elements.

Lemma 4.9. *Suppose M has no two disjoint 4-cocircuits and $|E(M)| \geq 10$. Let D_1, D_2, and D_3 be 4-cocircuits of M. If $|D_1 \cap D_2| = 2$, then $|D_3 \cap (D_1 \cup D_2)| \geq 2$.*

Proof. Suppose not. Then $|D_3 \cap (D_1 \cup D_2)| = 1$, and, more specifically, $|D_1 \cap D_2 \cap D_3| = 1$. Let $\{e\} = D_1 \cap D_2 \cap D_3$. By circuit elimination, there is a cocircuit $D_4 \subseteq (D_1 \cup D_2) - \{e\}$. As $D_3 \cap D_4 = \emptyset$, we see that $|D_4| \neq 4$. Therefore, $D_4 = (D_1 \cup D_2) - \{e\}$.

As $|E(M)| \geq 10$, we have $|E(M) - (D_1 \cup D_2 \cup D_3)| \geq 1$. Let $f \in E(M) - (D_1 \cup D_2 \cup D_3)$, and let C be a 4-circuit containing $\{e, f\}$. To preserve orthogonality, C must contain an element $g \in D_3 - \{e\}$ and an element $h \in (D_1 \cap D_2) - \{e\}$. But then $C = \{e, f, g, h\}$, and $|C \cap D_4| = 1$. This contradicts the orthogonality of C and D_4, proving the lemma. □

Concerning 4-cocircuits that meet in two elements, we may now prove that the two shared elements do not appear together in any other 4-cocircuits.

Lemma 4.10. *Suppose M has no two disjoint 4-cocircuits and $|E(M)| \geq 10$. Let D_1, D_2, and D_3 be distinct 4-cocircuits of M. If $D_1 \cap D_2 = \{x_1, x_2\}$, then $\{x_1, x_2\} \not\subseteq D_3$.*

Proof. Suppose not. Let $E(M) = \{x_1, x_2, \ldots, x_n\}$, where $n \geq 10$. Without loss of generality, $D_1 = \{x_1, x_2, x_3, x_4\}$, $D_2 = \{x_1, x_2, x_5, x_6\}$, and $D_3 = \{x_1, x_2, x_7, x_8\}$. Using circuit elimination on each pair in $\{D_1, D_2, D_3\}$ and eliminating x_2, we find that each of $\{x_1, x_3, x_4, x_5, x_6\}$, $\{x_1, x_3, x_4, x_7, x_8\}$, and $\{x_1, x_5, x_6, x_7, x_8\}$ contains a cocircuit. Each of those cocircuits must contain x_1, otherwise we get two disjoint 4-cocircuits. Further, each of these 5-element sets is, in fact, a cocircuit, as any of their 4-element subsets containing x_1 meets another 4-cocircuit in three elements. We will refer to these 5-cocircuits as D_5, D_6, and D_7, respectively.

Consider a 4-circuit, C_1, containing $\{x_1, x_9\}$. By considering the intersection of C_1 witheach of D_1, D_2, and D_3, we see that $x_2 \in C_1$. However, C_1 only meets each of D_5, D_6, and D_7 in a single element. By orthogonality, C_1 must have a second element in common with each of them. However, C_1 has only one additional element, and there is no single element other than x_1 that these 5-cocircuits have in common. This contradiction proves the lemma. □

This last preliminary lemma prohibits a particular configuration of 4-cocircuits.

Lemma 4.11. *Suppose M has no two disjoint 4-cocircuits and $|E(M)| \geq 10$. Let D_1, D_2, and D_3 be 4-cocircuits of M. If $|D_1 \cap D_2 \cap D_3| = 1$, then $|D_i \cap D_j| = 1$ for some pair $\{i, j\} \subseteq \{1, 2, 3\}$.*

Proof. Suppose not. Let $E(M) = \{x_1, x_2, \ldots, x_n\}$, where $n \geq 10$. By combining Proposition 4.2 and Lemma 4.10, we may let $D_1 = \{x_1, x_2, x_3, x_4\}$, $D_2 = \{x_1, x_2, x_5, x_6\}$, and $D_3 = \{x_1, x_3, x_5, x_7\}$, without loss of generality.

Consider a 4-circuit, C_1, containing $\{x_8, x_9\}$. If C_1 meets $D_1 \cup D_2 \cup D_3$, then it must do so in at least three elements to avoid an orthogonality contradiction; therefore, $C_1 \cap (D_1 \cup D_2 \cup D_3) = \emptyset$. We may assume $C_1 = \{x_8, x_9, x_{10}, x_{11}\}$.

Now consider a 4-cocircuit, D_4, containing x_8. Without loss of generality, $x_9 \in D_4$. By assumption, D_4 meets each of D_1, D_2, and D_3, and by Lemma 4.9, D_4 must contain two elements from each of $D_1 \cup D_2$, $D_1 \cup D_3$, and $D_2 \cup D_3$. If $x_1 \in D_4$, then, by Lemma 4.10, none of x_2, x_3, x_4, x_5, or x_6 is in D_4, a contradiction. Therefore, without loss of generality, $D_4 = \{x_2, x_3, x_8, x_9\}$.

Finally, consider a 4-circuit, C_2, containing $\{x_4, x_{10}\}$. It must meet $D_1 - \{x_4\}$ to avoid an orthogonality contradiction. If $x_1 \notin C_2$, then $x_2 \in C_2$, without loss of generality. This means $C_2 \cap D_2 \neq \emptyset$ and $C_2 \cap D_4 \neq \emptyset$. Since $D_2 - \{x_2\}$ has no element in common with $D_4 - \{x_2\}$, this is a contradiction to orthogonality. Thus $x_1 \in C_2$, and $C_2 \cap D_2 \neq \emptyset$ and $C_2 \cap D_3 \neq \emptyset$. Thus, $C_2 = \{x_1, x_4, x_5, x_{10}\}$. Similarly, a 4-circuit, C_3, containing $\{x_4, x_{11}\}$ must be $C_3 = \{x_1, x_4, x_5, x_{11}\}$. Then, $C_2 \cup C_3$ is a 5-point plane, and $\{x_4, x_5, x_{10}, x_{11}\}$ is a circuit that meets D_1 in a single element. This contradiction proves the lemma. $\square$

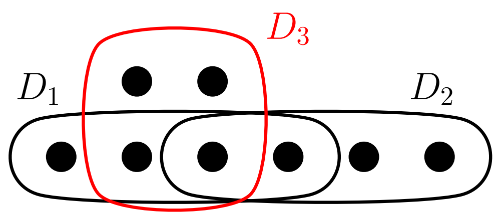
With those lemmas proved, we are ready to show that M must have two disjoint 4-cocircuits when it has at least 11 elements. This is the final proposition needed before proving our first main theorem. In the proof that follows, we will frequently refer to the way in which a 4-cocircuit meets two other 4-cocircuits that share two elements. For convenience, we introduce the following terminology.

4.12. Let D_1, D_2, and D_3 be 4-cocircuits of M, and suppose $|D_1 \cap D_2| = 2$. With respect to (D_1, D_2), we say that D_3 is of:

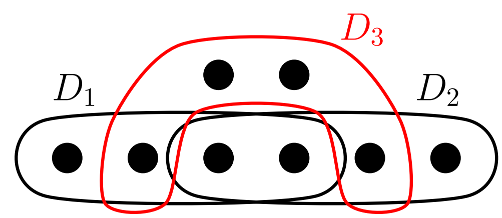
(i) *type-1* if $|D_3 \cap D_1 \cap D_2| = 1$, and $|D_3 \cap (D_1 - D_2)| = 1$, and $|D_3 \cap (D_2 - D_1)| = 0$;

(ii) *type-2* if $D_3 \cap D_1 \cap D_2 = \emptyset$, and $|D_3 \cap D_1| = |D_3 \cap D_2| = 1$; and

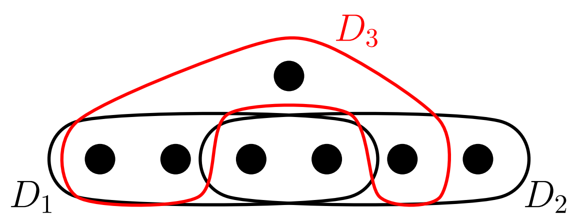
(iii) *type-3* if $D_3 \cap D_1 \cap D_2 = \emptyset$, and $|D_3 \cap D_1| = 2$, and $|D_3 \cap D_2| = 1$.



(a) (D_1, D_2)-Type-1 Configuration



(b) $\{D_1, D_2\}$-Type-2 Configuration



(c) (D_1, D_3)-Type-3 Configuration

Figure 4.2: Set diagrams of the structures described in 4.12.

Note that type-2 intersections are symmetric; therefore, we will denote this intersection by $\{D_1, D_2\}$-type-2. There will be occasions in which it is sufficient to specify that D_3 is either (D_1, D_2)-type-i or (D_2, D_1)-type-i, for $i \in \{1, 3\}$. In these instances, we will say that D_3 is $\{D_1, D_2\}$-type-i. The previous lemmas ensure that any 4-cocircuit not contained in such D_1 and D_2 must be one of the above types when M has no two disjoint 4-cocircuits and $|E(M)| \geq 11$. We prove this in the following lemma.

Lemma 4.13. *Suppose M has no two disjoint 4-cocircuits, $|E(M)| \geq 10$, and let D_1 and D_2 be 4-cocircuits of M such that $|D_1 \cap D_2| = 2$. If D_3 is a 4-cocircuit of M such that $D_3 \not\subseteq D_1 \cup D_2$, then D_3 is $\{D_1, D_2\}$-type-i, for exactly one $i \in \{1, 2, 3\}$.*

Proof. By Lemma 4.10, either $D_1 \cap D_2 \cap D_3 = \emptyset$ or $|D_1 \cap D_2 \cap D_3| = 1$.

Suppose, first, that $|D_1 \cap D_2 \cap D_3| = 1$. In this case, $|D_3 \cap (D_1 \cup D_2)| \geq 2$ by Lemma 4.9, so we may assume $D_3 \cap (D_1 - D_2) \neq \emptyset$. By Lemma 4.11, we know $|D_i \cap D_j| = 1$ for some $\{i, j\} \subseteq \{1, 2, 3\}$, and, since $|D_1 \cap D_2| = 2$ and $|D_1 \cap D_3| = 2$, it must be that $|D_2 \cap D_3| = 1$. Therefore, $|D_3 \cap (D_2 - D_1)| = 0$, and D_3 is (D_1, D_2)-type-1.

Suppose, instead, that $D_1 \cap D_2 \cap D_3 = \emptyset$. As M has no two disjoint 4-cocircuits, we have $D_1 \cap D_3 \neq \emptyset$ and $D_2 \cap D_3 \neq \emptyset$. By Lemma 4.11, it cannot be that $|D_1 \cap D_3| = |D_2 \cap D_3| = 2$, so either $|D_1 \cap D_3| = |D_2 \cap D_3| = 1$, or $|D_1 \cap D_3| = 2$ and $|D_2 \cap D_3| = 1$, without loss of generality. These cases yield the $\{D_1, D_2\}$-type-2 and (D_1, D_2)-type-3 configurations, respectively. $\square$

Now that we have narrowed down the possible configurations of 4-cocircuit intersections, we will systematically prove that each of these configurations cannot occur when $|E(M)| \geq 11$ unless M has two disjoint 4-cocircuits. This proof if very technical, and will be divided into three parts, with each part addressing one of the 4-cocircuit configurations. Parts one and two will be considered lemmas, and the final part will be the central proposition of this section. Throughout, we assume that $E(M) = \{x_1, x_2, \ldots, x_n\}$, and D_1 and D_2 are 4-cocircuits of M such that $D_1 = \{x_1, x_2, x_3, x_4\}$ and $D_2 = \{x_1, x_2, x_5, x_6\}$.

Lemma 4.14. *Suppose M has no two disjoint 4-cocircuits, $|E(M)| \geq 11$, and let D_1 and D_2 be 4-cocircuits of M such that $|D_1 \cap D_2| = 2$. If D_3 is another 4-cocircuit of M, then D_3 is not $\{D_1, D_2\}$-type-2.*

Proof. Suppose D_3 is $\{D_1, D_2\}$-type-2. Without loss of generality, $D_3 = \{x_3, x_5, x_7, x_8\}$. By $(P2)$, we are guaranteed a 4-cocircuit D_4 containing x_9. Before determining the rest of the elements in D_4, we will prove that

Claim 4.14.1. *D_4 and all further 4-cocircuits must be $\{D_1, D_2\}$-type-1.*

If D_4 is not $\{D_1, D_2\}$-type-1, then, by Lemma 4.12, it must be either $\{D_1, D_2\}$-type-2 or $\{D_1, D_2\}$-type-3. We treat the second case first.

Claim 4.14.1.1. *D_4 is not $\{D_1, D_2\}$-type-3.*

Assume the contrary. Then either $D_4 = \{x_3, x_4, x_5, x_9\}$ or $D_4 = \{x_3, x_4, x_6, x_9\}$. In the former case we have $|D_3 \cap D_4| = 2$ and $|D_2 \cap (D_3 \cup D_4)| < 2$, contradicting Lemma 4.9. Similarly, in the latter case, we have $|D_1 \cap D_4| = 2$ and $|D_3 \cap (D_1 \cup D_4)| < 2$, again contradicting Lemma 4.9. This completes the argument, proving that D_4 and all further 4-cocircuits in Claim 4.14.1 cannot be $\{D_1, D_2\}$-type-3.

Claim 4.14.1.2. *D_4 is not $\{D_1, D_2\}$-type-2.*

Assume, instead, that D_4 is $\{D_1, D_2\}$-type-2. Then $|D_4 \cap \{x_3, x_5\}| \leq 1$; otherwise, if $\{x_3, x_5\} \subseteq D_4$, then $|D_1 \cap (D_3 \cup D_4)| < 2$ and Lemma 4.9 provides a contradiction.

Suppose $|D_4 \cap \{x_3, x_5\}| = 1$. Then, without loss of generality, $x_3 \in D_4$. Since D_4 is $\{D_1, D_2\}$-type-2, this implies $x_6 \in D_4$. Further, one of x_7 or x_8 must be in D_4, otherwise $|D_1 \cap D_3 \cap D_4| = 1$ and Lemma 4.7 provides a contradiction. Hence, without loss of generality, $D_4 = \{x_3, x_6, x_7, x_9\}$, but now $|D_3 \cap D_4| = 2$ and $|D_1 \cap (D_3 \cup D_4)| < 2$, a contradiction to Lemma 4.9.

We now know that D_4 avoids $\{x_3, x_5\}$. This means we may assume that $D_4 = \{x_4, x_6, x_7, x_9\}$. By $(P2)$, we have a 4-cocircuit D_5 containing x_{10}. By 4.14.1.1, D_5 cannot be $\{D_1, D_2\}$-type-3. If D_5 is $\{D_1, D_2\}$-type-2, then, by the above analysis, $\{x_4, x_6\} \subseteq D_5$ and $\{x_7, x_8\} \cap D_5 \neq \emptyset$. If $D_5 = \{x_4, x_6, x_7, x_{10}\}$, then $|D_4 \cap D_5| = 3$ and Proposition 4.2 provides a contradiction. If $D_5 = \{x_4, x_6, x_8, x_{10}\}$, then $|D_1 \cap (D_4 \cup D_5)|$ and Lemma 4.9 provides a contradiction. Therefore D_5 must be $\{D_1, D_2\}$-type-1. By symmetry, demonstrated in Figure 4.3, we may assume that D_5 is (D_1, D_2)-type-1.

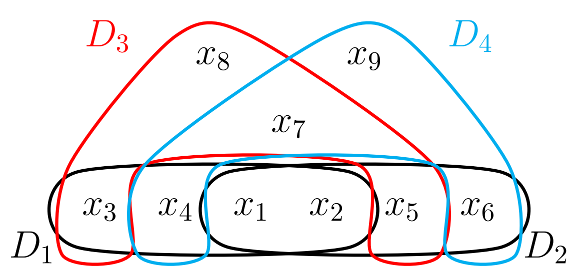


Figure 4.3: Set diagram of the symmetry in 4.14.1.2.

Hence, without loss of generality, $\{x_1, x_3\} \subseteq D_5$. Further, the remaining element of D_5 must be either x_7 or x_9, otherwise $D_4 \cap D_5 = \emptyset$. Each of these leads to contradiction. If $D_5 = \{x_3, x_5, x_7, x_{10}\}$, then $|D_5 \cap D_3| = 2$ and $|D_4 \cap (D_5 \cup D_3)| < 2$, contradicting Lemma 4.9. Similarly, if $D_5 = \{x_3, x_5, x_9, x_{10}\}$, then $|D_5 \cap D_1| = 2$ and $|D_3 \cap (D_5 \cup D_1)| < 2$, again contradicting Lemma 4.9. This completes the proof of Claim 4.14.1.

The following claim is an immediate corollary of the previous claim. In fact, it is merely a generalized restatement presented here for ease of reference.

Claim 4.14.2. *If D_i, D_j, D_k, and D_l are 4-cocircuits of M such that $|D_i \cap D_j| = 2$ and D_k is $\{D_i, D_j\}$-type-2, then D_l is $\{D_i, D_j\}$-type-1.*

By 4.14.1, D_4 is $\{D_1, D_2\}$-type-1. We may assume without loss of generality that D_4 is (D_1, D_2)-type-1; that is, D_4 meets $\{x_3, x_4\}$ but avoids $\{x_5, x_6\}$. It follows that, since $|D_1 \cap D_4| = 2$, we have $D_4 \cap \{x_7, x_8\} \neq \emptyset$, otherwise $|D_3 \cap (D_1 \cup D_4)| < 2$, contradicting Lemma 4.9. Hence, either $D_4 = \{x_1, x_3, x_7, x_9\}$ or $D_4 = \{x_1, x_4, x_7, x_9\}$, without loss of generality. We will first show that

Claim 4.14.3. $D_4 \neq \{x_1, x_3, x_7, x_9\}$.

Assume the contrary, in which case D_1, D_2, D_3, and D_4 are as in Figure 4.4. Now $|D_3 \cap D_4| = 2$, and D_2 is (D_3, D_4)-type-2. Therefore Claim 4.14.2 implies that all further 4-cocircuits must be $\{D_3, D_4\}$-type-1. Moreover, just as D_4 necessarily meets $\{x_7, x_8\}$, so must all further cocircuits meet both $\{x_7, x_8\}$ and $\{x_2, x_6\}$. This is because the structure given by D_1, D_2, and D_3 that forced one of x_7 and x_8 in D_4 is now present in D_1, D_2, and D_4, as demonstrated in the symmetry about a vertical line through x_5 in Figure 4.4.

By $(P2)$, there is a 4-cocircuit D_5 containing x_{10}. In order to be both $\{D_1, D_2\}$-type-1 and $\{D_3, D_4\}$-type-1, D_5 must contain exactly one element from each of the following: $D_1 \cap D_2 = \{x_1, x_2\}$, $D_1 \bigtriangleup D_2 = \{x_3, x_4, x_5, x_6\}$, $D_3 \cap D_4 = \{x_3, x_7\}$, and $D_3 \bigtriangleup D_4 = \{x_1, x_5, x_8, x_9\}$.

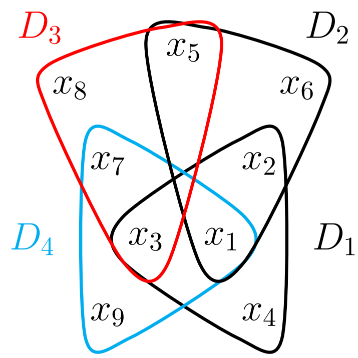


Figure 4.4: Set diagram of the structure of cocircuits in Claim 4.14.3.

Note that we are not asserting that either $D_1 \triangle D_4$ or $D_3 \triangle D_4$ is a cocircuit. Because $D_1 \cap D_4 = \{x_1, x_3\}$, Lemma 4.10 implies $\{x_1, x_3\} \nsubseteq D_5$.

Claim 4.14.3.1. $x_1 \in D_5$.

Assume the contrary. By the symmetry demonstrated in Figure 4.4, we may assume that $x_3 \notin D_5$. This implies $\{x_2, x_7\} \subseteq D_5$, and therefore $\{x_5\} = (D_1 \triangle D_2) \cap (D_3 \triangle D_4) \subseteq D_5$. Hence $D_5 = \{x_2, x_5, x_7, x_{10}\}$. Note that the vertical symmetry in Figure 4.4 still holds with the inclusion of D_5. Now we have two new pairs of 4-cocircuits that meet in two elements; namely, $\{D_2, D_5\}$ and $\{D_3, D_5\}$. Further, D_4 is $\{D_2, D_5\}$-type-2, D_1 is $\{D_3, D_5\}$-type-2, and D_5 is $\{D_1, D_4\}$-type-2. Therefore, by Claim 4.14.2, any further 4-cocircuits must also contain exactly one element from each intersection and from each symmetric difference of these pairs. By property $(P2)$, there is a 4-cocircuit D_6 containing x_{11}. By Lemma 4.10, if $x_1 \in D_6$, then $\{x_2\} = (D_1 \cap D_2) - x_1 \nsubseteq D_6$ and $\{x_3\} = (D_1 \cap D_4) - x_1 \nsubseteq D_6$. This implies $\{x_5, x_7\} \subseteq D_6$, since $\{x_5\} = (D_2 \cap D_5) - x_2$ and $\{x_7\} = (D_3 \cap D_4) - x_3$. This, however, is a contradiction by Lemma 4.10, since $\{x_5, x_7\} = D_3 \cap D_5$. Thus $x_1 \notin D_6$, and, by the aforementioned symmetry, this implies $x_3 \notin D_6$. However, then D_6 does not contain an element from $D_1 \cap D_4$. This contradicts Claim 4.14.2, and this case cannot arise.

By the symmetry noted above, we may assume that one of x_1 or x_3 is in every further 4-cocircuit of M, otherwise we get a contradiction as in 4.14.3.1. Therefore, we may assume $x_1 \in D_5$. Hence, $x_7 \in D_5$ by Claim 4.14.2, since $\{x_7\} = (D_3 \cap D_4) - x_3$. Note that $x_5 \notin D_5$

by Lemma 4.13 with respect to $\{D_3, D_4\}$. Therefore, the remaining element of D_5 must be one of x_4 and x_6, as $D_5 \cap (D_1 \triangle D_2) \neq \emptyset$.

Assume, first, that $x_4 \in D_5$. In this case, $|D_1 \cap D_4| = 2$ and D_3 is $\{D_1, D_4\}$-type-2. Given $(P2)$, there must be a 4-cocircuit D_6 containing x_{11}. By Claim 4.14.2, D_6 must contain exactly one element from each of $\{x_1, x_2\} = D_1 \cap D_2$, $\{x_1, x_4\} = D_1 \cap D_5$, and $\{x_3, x_7\} = D_3 \cap D_4$, as each of these pairs has a 4-cocircuit that is type-2 with respect to them. Further, by Claim 4.14.3.1, we know one of x_1 and x_3 is in D_6. If $x_1 \notin D_6$, then $\{x_2, x_3, x_4\} \subseteq D_6$, and Proposition 4.2 provides a contradiction. If $x_1 \in D_6$, then $\{x_2, x_3, x_4\} \cap D_6 = \emptyset$, and so $x_7 \in D_6$. But $\{x_1, x_7\} = D_4 \cap D_5$, and so we contradict Lemma 4.10. Therefore $x_4 \notin D_5$.

This implies that $D_5 = \{x_1, x_6, x_7, x_{10}\}$. Again, take a 4-cocircuit D_6 containing x_{11}, and consider its remaining elements. We know that one of x_1 and x_3 is in D_6. If $x_1 \in D_6$, then by the previous argument concerning D_5, now applied to D_6, we get $D_6 = \{x_1, x_6, x_7, x_{11}\}$, which contradicts Proposition 4.2. Therefore, $x_1 \notin D_6$, and $x_3 \in D_6$. Observe that $|D_2 \cap D_5| = 2$, and D_3 is $\{D_2, D_5\}$-type-2. By Claim 4.14.2, D_6 must contain exactly one element from each of $\{x_1, x_2\} = D_1 \cap D_2$, $\{x_1, x_6\} = D_2 \cap D_5$, and $\{x_1, x_5, x_8, x_9\}$. This is impossible, as D_6 has only four elements. Thus this case cannot arise and Claim 4.14.3 holds.

Restating Claim 4.14.3 more generally, we have the following. This is seen by replacing D_1, D_2, D_3, and D_4 with D_i, D_j, D_k, and D_l.

Claim 4.14.4. *If D_i, D_j, D_k, and D_l are 4-cocircuits of M such that $|D_i \cap D_j| = 2$ and D_k is $\{D_i, D_j\}$-type-2, then $D_l \cap D_i \cap D_k = \emptyset$ and $D_l \cap D_j \cap D_k = \emptyset$.*

The only remaining case is when $D_4 = \{x_1, x_4, x_7, x_9\}$. We illustrate this arrangement of cocircuits in Figure 4.5. Now we have $|D_1 \cap D_4| = 2$ and D_3 is $\{D_1, D_4\}$-type-2. Therefore, by Claim 4.14.2, further 4-cocircuits must be both $\{D_1, D_2\}$-type-1 and $\{D_1, D_4\}$-type-1.

Let D_5 be a 4-cocircuit containing x_{10}. We first prove that $x_1 \in D_5$, via the following claim.

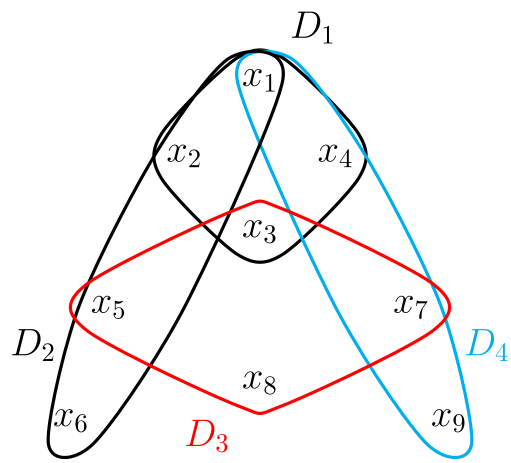


Figure 4.5: Set diagram of remaining possible structure of cocircuits.

Claim 4.14.5. *There is a 4-cocircuit containing x_1 and some element not in $\{x_1, x_2, \ldots, x_9\}$.*

Suppose not. Then $\{x_2, x_4\} \subseteq D_5$. Further, as M has no disjoint 4-cocircuits, $D_5 \cap D_3 \neq \emptyset$. Since D_5 is both $\{D_1, D_2\}$-type-1 and $\{D_1, D_4\}$-type-1, we have $\{x_3, x_5, x_7\} \cap D_5 = \emptyset$. Therefore, $D_5 = \{x_2, x_4, x_8, x_{10}\}$. By $(P2)$, we have a 4-cocircuit D_6 containing x_{11}. However, by the same reasoning, we get $\{x_2, x_4\} \subseteq D_6$. This is a contradiction to Lemma 4.10, since $D_1 \cap D_5 = \{x_2, x_4\}$. Thus the claim holds.

We may assume, then, that $x_1 \in D_5$. Now,

Claim 4.14.6. *exactly one element from each $\{x_3, x_5, x_6\}$ and $\{x_3, x_7, x_9\}$ is in D_5.*

This is because D_5 must be either type-1, type-2, or type-3 with respect to the pairs $\{D_1, D_2\}$ and $\{D_1, D_4\}$, by Lemma 4.13.

Claim 4.14.7. *Any 4-cocircuit containing an element not in $\{x_1, x_2, \ldots, x_9\}$ does not contain both x_1 and x_3.*

If $x_3 \in D_5$, then $\{x_2, x_4, x_5, x_6, x_7, x_9\} \cap D_5 = \emptyset$ by Claim 4.14.2. In this case, $|D_1 \cap D_5| = 2$, which mandates that D_3 is one $\{D_1, D_5\}$-type-1, $\{D_1, D_5\}$-type-2, or $\{D_1, D_5\}$-type-3, by Lemma 4.12. The only possibility is that D_3 is (D_5, D_1)-type-1, which requires $x_8 \in D_5$. Therefore, $D_5 = \{x_1, x_3, x_8, x_{10}\}$. Now $(P2)$ guarantees a 4-cocircuit D_6 containing x_{11}. Note that $|D_3 \cap D_5| = 2$ and D_4 is $\{D_3, D_5\}$-type-2, so, in addition to the previous restrictions dictated by Claim 4.14.2, namely, that D_6 must contain exactly one element from each intersection and symmetric difference of the pairs $\{D_1, D_2\}$ and $\{D_1, D_4\}$, we have that D_6 must contain exactly one element from each $\{x_3, x_8\} = D_3 \cap D_5$ and $\{x_1, x_5, x_7, x_{10}\} = D_3 \triangle D_5$. Therefore, we must have $x_1 \in D_6$, which precludes any of x_2, x_4, x_5, x_7, or x_{10} from being members of D_6. This forces $x_3 \in D_6$, otherwise D_6 will miss either $D_1 \triangle D_2$ or $D_3 \cap D_5$, but this contradicts Lemma 4.10, since $D_1 \cap D_5 = \{x_1, x_3\}$. Thus $x_3 \notin D_5$. This proves the claim by the left-right symmetry between x_1 and x_3, pictured in Figure 4.5.

Since D_5 must be both $\{D_1, D_2\}$-type-1 and $\{D_1, D_4\}$-type-1, but does not contain x_3, it must contain exactly one element from each $\{x_5, x_6\}$ and $\{x_7, x_9\}$, by 4.14.6. Further, D_5 must contain at least one of x_5 or x_7, otherwise it is disjoint from D_3. By the symmetry about the vertical line through x_1 in Figure 4.5, we may assume $x_5 \in D_5$. Then $|D_2 \cap D_5| = 2$, and D_3 must be one of $\{D_2, D_5\}$-type-1, $\{D_2, D_5\}$-type-2, or $\{D_2, D_5\}$-type-3, forcing x_7 in D_5. Now, $(P2)$ guarantees a 4-cocircuit D_6 with x_{11}, but, by symmetry, a similar argument as the one just applied to D_5 forces $\{x_5, x_7\} \subseteq D_6$. This contradicts Lemma 4.10, as $D_3 \cap D_5 = \{x_3, x_5\}$. Thus we are unable to find viable cocircuits to cover the elements of M, and the lemma is proved. □

Lemma 4.15. *Suppose M has no two disjoint 4-cocircuits, $|E(M)| \geq 11$, and let D_1 and D_2 be 4-cocircuits of M such that $|D_1 \cap D_2| = 2$. If D_3 is another 4-cocircuit of M, then D_3 is not $\{D_1, D_2\}$-type-3.*

Proof. Let $E(M) = \{x_1, x_2, \ldots, x_n\}$, and suppose the contrary, that D_3 is $\{D_1, D_2\}$-type-3. Without loss of generality, we may assume $D_3 = \{x_3, x_4, x_5, x_7\}$. Note that D_2 is (D_1, D_3)-type-3.

Claim 4.15.1. *There are no further $\{D_1, D_i\}$-type-3 4-cocircuits, for $i \in \{2, 3\}$.*

By symmetry, it suffices to show that no further 4-cocircuits are (D_1, D_2)-type-3 or (D_2, D_1)-type-3. By $(P2)$, we have a 4-cocircuit D_4 containing x_8. If D_4 is (D_1, D_2)-type-3, then $\{x_3, x_4\} \subseteq D_4$, contradicting Lemma 4.10.

Suppose, then, that D_4 is (D_2, D_1)-type-3. Without loss of generality, $D_4 = \{x_3, x_5, x_6, x_8\}$, as depicted in Figure 4.6.

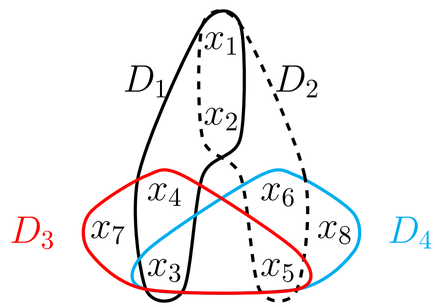


Figure 4.6: Set diagram of the 4-cocircuits in Claim 4.15.1.

By $(P2)$, there must be a 4-cocircuit D_5 containing x_9. By Lemma 4.10, D_5 cannot be $\{D_1, D_2\}$-type-3. Therefore, by Lemma 4.12 and Lemma 4.14, D_5 must be of $\{D_1, D_2\}$-type-1. By symmetry, we may assume $x_1 \in D_5$ and, further, that D_5 contains exactly one element from $\{x_3, x_4\}$ and is disjoint from $\{x_5, x_6\}$. If $x_4 \in D_5$, then x_8 must also be in D_5, otherwise $D_4 \cap D_5 = \emptyset$. However, in this case, D_5 is $\{D_3, D_4\}$-type-2 and Lemma 4.14 provides a contradiction.

Therefore, we may assume that $x_3 \in D_5$. Now $x_8 \in D_5$, otherwise D_5 is $\{D_2, D_4\}$-type-2. Hence $D_5 = \{x_1, x_3, x_8, x_9\}$. As M has $(P2)$, there must be a 4-cocircuit D_6 containing x_{10}. By similar reasoning to above, D_6 must contain either x_1 or x_2, and either $\{x_3, x_8\}$ or $\{x_5, x_7\}$. Lemma 4.10 indicates that $\{x_5, x_7\} \subseteq D_6$. There must be a further 4-cocircuit D_7 that contains $\{x_{11}\}$. However, now D_7 cannot contain either $\{x_3, x_8\}$ or $\{x_5, x_7\}$ without violating Lemma 4.10. This contradiction completes the proof of Claim 4.15.1.

Observe that Claim 4.15.1 can be restated more generally, as follows:

Claim 4.15.2. *Given 4-circuits D_i, D_j, D_k, and D_l such that $|D_i \cap D_j| = |D_i \cap D_k| = 2$, if D_k is (D_i, D_j)-type-3, then D_l is neither $\{D_i, D_j\}$-type-3 nor $\{D_i, D_k\}$-type-3.*

Now, every 4-cocircuit containing an element not in $\{x_1, x_2, \ldots, x_7\}$ must be both $\{D_1, D_2\}$-type-1 and $\{D_1, D_3\}$-type-1. We restrict this even further by proving the following:

Claim 4.15.3. *All 4-cocircuits containing an element not in $\{x_1, x_2, \ldots, x_7\}$ must be both (D_1, D_2)-type-1 and (D_1, D_3)-type-1.*

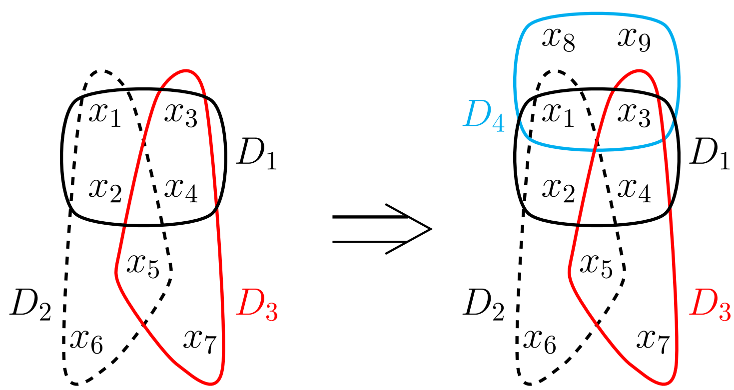


Figure 4.7: A set diagram of the 4-cocircuits structure after Claim 4.15.3

.

Note that any 4-cocircuit that is (D_1, D_2)-type-1 is also (D_1, D_3)-type-1. Let D_4 be a 4-cocircuit containing x_8, and suppose D_4 is neither (D_1, D_2)-type-1 nor (D_1, D_3)-type-1. If $x_5 \in D_4$, then $D_4 = \{x_1, x_3, x_5, x_8\}$, without loss of generality. However, now $|D_1 \cap D_2 \cap D_4| = 1$ and $|D_1 \cap D_2| = |D_1 \cap D_4| = |D_2 \cap D_4| = 2$, and Lemma 4.11 provides a contradiction.

We now know that, $x_5 \notin D_4$, and so $\{x_6, x_7\} \subseteq D_4$. By assumption, D_4 must meet D_1. Without loss of generality, we may assume $x_1 \in D_4$, but this gives a contradiction to Lemma 4.14, as now D_4 is $\{D_1, D_3\}$-type-2. Thus Claim 4.15.3 is proved.

By $(P2)$, there is a 4-cocircuit D_5 containing x_{10}. We know now that D_5 must be both (D_1, D_2)-type-1 and (D_1, D_3)-type-1. This implies that D_5 meets each of $\{x_1, x_2\}$ and $\{x_3, x_4\}$ in exactly one element, and is disjoint from $\{x_5, x_6, x_7\}$. By Lemma 4.10, we know $\{x_1, x_3\} \not\subseteq D_5$, and so, by symmetry, we have that D_5 contains either $\{x_1, x_4\}$ or $\{x_2, x_4\}$.

If $\{x_1, x_4\} \subseteq D_5$, then $\{x_8, x_9\} \cap D_5 \neq \emptyset$, otherwise $|D_2 \cap D_4 \cap D_5| = 1$ and $|D_2 \cap D_4| = |D_2 \cap D_5| = |D_4 \cap D_5| = 1$, and Lemma 4.7 provides a contradiction. Therefore, we may assume that $D_5 = \{x_1, x_4, x_8, x_{10}\}$; however, now $|D_1 \cap D_4 \cap D_5| = 1$ and $|D_1 \cap D_4| = |D_1 \cap D_5| = |D_4 \cap D_5| = 2$, and Lemma 4.11 provides a contradiction.

Therefore, it must be that $\{x_2, x_4\} \subseteq D_5$. Now, $D_5 \cap \{x_8, x_9\} \neq \emptyset$, otherwise D_4 and D_5 are disjoint. Hence, $D_5 = \{x_2, x_4, x_8, x_{10}\}$, without loss of generality.

There must be a 4-cocircuit D_6 containing $\{x_{11}\}$. As the restrictions on D_5 apply similarly to D_6, we may immediately conclude that $\{x_2, x_4\} \subseteq D_6$, which contradicts Lemma 4.10. Thus, no such D_6 may exist, and the proof of the lemma is complete. □

Proposition 4.16. *If $|E(M)| \geq 11$, then M has two disjoint 4-cocircuits.*

Proof. Let $E(M) = \{x_1, x_2, \ldots, x_n\}$, and suppose that M has no two disjoint 4-cocircuits. By Lemma 4.8, we have 4-cocircuits D_1 and D_2 meeting in two elements. Without loss of generality, let $D_1 = \{x_1, x_2, x_3, x_4\}$ and $D_2 = \{x_1, x_2, x_5, x_6\}$. Lemma 4.14 and Lemma 4.15 indicate that all further 4-cocircuits of M must be $\{D_1, D_2\}$-type-1. By $(P2)$, there is a 4-cocircuit D_3 containing x_7. Without loss of generality, $D_3 = \{x_1, x_3, x_7, x_8\}$. Since $|D_1 \cap D_3| = 2$, further 4-cocircuits must also be $\{D_1, D_3\}$-type-1. By $(P2)$, there is a 4-cocircuit D_4 containing x_9.

Claim 4.16.1. *D_4 is not both (D_2, D_1)-type-1 and (D_3, D_1)-type-1.*

If D_4 is both (D_2, D_1)-type-1 and (D_3, D_1)-type-1, then $\{x_5, x_7\} \subseteq D_4$, without loss of generality. As D_4 must also contain an element from both $D_1 \cap D_2$ and $D_1 \cap D_3$, we have $D_4 = \{x_1, x_5, x_7, x_9\}$. See Figure 4.8 for reference.

By $(P2)$, there is a 4-cocircuit D_5 containing x_{10} which must be $\{D_1, D_2\}$-type-1, $\{D_1, D_3\}$-type-1, $\{D_2, D_4\}$-type-1, and $\{D_3, D_4\}$-type-1. This forces x_1 into D_5, and further requires either $\{x_4, x_9\}$ or $\{x_6, x_8\}$ to be contained in D_5. These two possibilities are equivalent by symmetry, observed by rotating the second configuration in Figure 4.8 , so we may assume

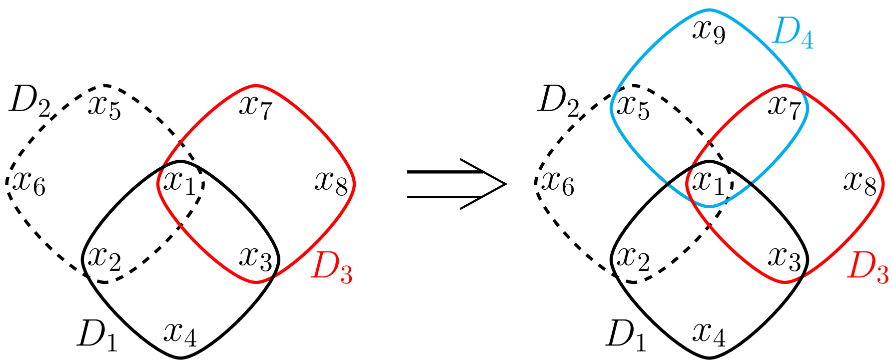


Figure 4.8: A set diagram of the 4-cocircuits in Claim 4.16.1

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$D_5 = \{x_1, x_4, x_9, x_{10}\}$. But now $|D_2 \cap D_3 \cap D_5| = 1$ and $|D_2 \cap D_3| = |D_2 \cap D_5| = |D_3 \cap D_5| = 1$. Since $|E(M)| > 10$, this is a contradiction by Lemma 4.7. This proves Claim 4.16.1.

Now we may assume that D_4 is (D_1, D_2)-type-1. Since D_4 is $\{D_1, D_3\}$-type-1, it must contain exactly one element from $D_1 \cap D_3 = \{x_1, x_3\}$. If $x_1 \in D_4$, then $x_4 \in D_4$, and we may assume $D_4 = \{x_1, x_4, x_9, x_{10}\}$; however, now $|D_2 \cap D_3 \cap D_4| = 1$ and $|D_2 \cap D_3| = |D_2 \cap D_4| = |D_3 \cap D_4| = 1$, and so Lemma 4.7 provids a contradiction.

Hence, $x_1 \notin D_4$. There D_4 must contain both x_2 and x_3 and no other elements of $D_1 \cup D_2 \cup D_3$. Without loss of generality, $D_4 = \{x_2, x_3, x_9, x_{10}\}$. There must be a 4-cocircuit D_6 containing x_{11}. As the above analysis applies to D_6, it follows that $\{x_2, x_3\} \subseteq D_6$. This is a contradiction by Lemma 4.10, as $D_1 \cap D_5 = \{x_2, x_3\}$. We have eliminated all possible types of 4-cocircuit intersection with $D_1 \cup D_2$. Thus there must be at least two disjoint 4-cocircuits in M, and the proof is complete.

□

We conclude this section by extending the previous proposition to a statement for matroids on thirteen elements. This result comes in three parts. First, we show that three pairwise-disjoint 4-cocircuits form a local $M(K_{3,4})$-structure. Then, we determine that, when M has exactly twelve elements, there is a specific configuration of 4-cocircuits that arises. Finally, we prove that, when M has at least thirteen elements, it must also have three pairwise-disjoint 4-cocircuits.

Lemma 4.17. *If D_1, D_2, and D_3 are pairwise-disjoint 4-cocircuits of M, then $M|(D_1 \cup D_2 \cup D_3) \cong M(K_{3,4})$.*

Proof. Let $E(M) = \{x_1, x_2, \ldots, x_n\}$. By Proposition 4.6, we know $M|(D_i \cup D_j) \cong M(K_{2,4})$, for $\{i, j\} \subseteq \{1, 2, 3\}$. Without loss of generality, let $D_1 = \{x_1, x_2, x_3, x_4\}$, $D_2 = \{x_5, x_6, x_7, x_8\}$, $D_3 = \{x_9, x_{10}, x_{11}, x_{12}\}$, and $\{x_1, x_5\}$, $\{x_2, x_6\}$, $\{x_3, x_7\}$, and $\{x_4, x_8\}$ be the series pairs in $M|(D_1 \cup D_2)$. The elements in these pairs are always found together in circuits contained in $D_1 \cup D_2$. The elements of D_3 appear together in pairs in 4-circuits contained in $D_1 \cup D_3$ and $D_2 \cup D_3$, again by the $M(K_{2,4})$ structure given by Proposition 4.6. We will show that these pairs correspond with the pairs that form the 4-circuits of $D_1 \cup D_2$; that is, we show that if x_i and x_j always appear together in the 4-circuits contained in $D_1 \cup D_2$, and x_i and x_k always appear together in the 4-circuits contained in $D_1 \cup D_3$, then x_j and x_k always appear together in the 4-circuits contained in $D_2 \cup D_3$. Without loss of generality, suppose $\{x_1, x_9\}$, $\{x_2, x_{10}\}$, $\{x_3, x_{11}\}$, and $\{x_4, x_{12}\}$ always appear together in the 4-circuits containined in $D_1 \cup D_3$, and compare this with Figure 4.9.

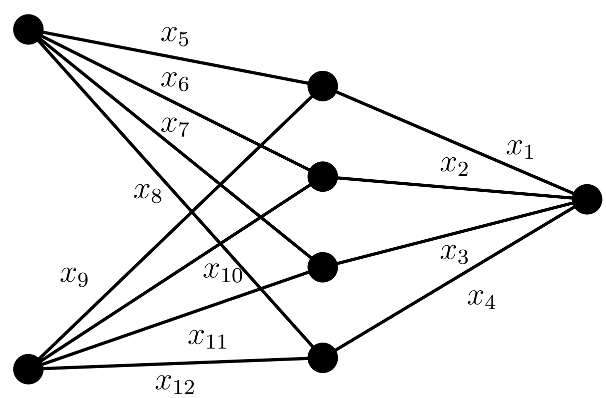

Figure 4.9: $K_{3,4}$ with labeled edges.

By circuit elimination, there is a circuit, C, contained in $(\{x_1, x_2, x_5, x_6\} \cup \{x_1, x_2, x_9, x_{10}\}) - \{x_1\}$. By orthogonality, $C = \{x_5, x_6, x_9, x_{10}\}$. By repeating this argument, we find that the pairs $\{x_5, x_9\}$, $\{x_6, x_{10}\}$, $\{x_7, x_{11}\}$, and $\{x_8, x_{12}\}$ always appear together in 4-circuits contained in $D_2 \cup D_3$.

Consider $M' = M(K_{3,4})$ on the ground set $\{x_1, x_2, ..., x_{12}\} = X$, say, where $K_{3,4}$ is labeled as in Figure 4.9. Given that $M|X$ is connected, Theorem 3.6 indicates that if $r(M') = r(M|X)$ and the identity map is weak map from M' to $M|X$, then $M' = M|X$.

First we check the rank. Evidently, $r(M|X) = r_M(X - D_1) + 1 = 6 = r(M')$, as desired. We will show that each circuit of M' is a dependent set in $M|X$. The 4-circuits are identical, so all that remains is to check the 6-circuits. Let C' be a 6-circuit of M'. Without loss of generality, we may assume $C' = \{x_1, x_3, x_5, x_6, x_{10}, x_{11}\}$. We know that $C_1 = \{x_1, x_2, x_5, x_6\}$ and $C_2 = \{x_2, x_3, x_{10}, x_{11}\}$ are circuits in $M|X$. Therefore, there is a circuit in $M|X$ contained in $(C_1 \cup C_2) - x_2$, and so $\{x_1, x_3, x_5, x_6, x_{10}, x_{11}\}$ is dependent in $M|X$. Thus the identity map is a weak map from M' to $M|X$. Hence $M' = M|X$. $\square$

The configuration of 4-cocircuits described in the following lemma is depicted in Figure 4.10. Note that elements in 4-cocircuits are contained in an oval, and elements in local series pair are connected by a green line segment.

Lemma 4.18. *If $|E(M)| = 12$, then M has four 4-cocircuits $D_1, D_2, D_3,$ and D_4 such that $D_1 \cap D_2 = D_3 \cap D_4 = \emptyset$ and $|D_i \cap D_j| = 1$ for $i \in \{1, 2\}$ and $j \in \{3, 4\}$.*

Proof. Let $E(M) = \{x_1, x_2, \ldots, x_n\}$. We know M has two disjoint 4-cocircuits, D_1 and D_2 by Proposition 4.16. Moreover, $M|(D_1 \cup D_2) \cong M(K_{2,4})$ by Proposition 4.6. Without loss of generality, let $D_1 = \{x_1, x_2, x_3, x_4\}$, $D_2 = \{x_5, x_6, x_7, x_8\}$, and $\{x_1, x_5\}$, $\{x_2, x_6\}$, $\{x_3, x_7\}$, and $\{x_4, x_8\}$ be the series pairs in $M|(D_1 \cup D_2)$. The elements in these pairs are always found together in circuits contained in $D_1 \cup D_2$. By assumption, x_9 is in a 4-cocircuit, say D_3. If D_3 is disjoint from both D_1 and D_2, then, by Lemma 4.17, we know $M \cong M(K_{3,4})$. This is a contradiction, as $M(K_{3,4})$ is not 4-connected. Therefore, we may assume that D_3 meets $D_1 \cup D_2$. By orthogonality with the 4-circuits contained in $M|(D_1 \cup D_2)$, we see that $D_3 \cap (D_1 \cup D_2)$ must be one of the aforementioned pairs. Without loss of generality, $D_3 = \{x_1, x_5, x_9, x_{10}\}$. Let D_4 be a 4-cocircuit containing x_{11}. As with D_3, we know that D_4

must meet $D_1 \cup D_2$ in one of the aforementioned series pairs. Therefore, if $D_3 \cup D_4 = \emptyset$, then we are done. Hence, assume the contrary.

Suppose $x_1 \in D_4$. Then we may assume that $D_4 = \{x_1, x_5, x_{11}, x_{12}\}$. Now $(D_3 \cup D_4) - x_1$ contains a cocircuit and, by orthogonality, this cocircuit avoids x_5. Hence $\{x_9, x_{10}, x_{11}, x_{12}\}$ is a cocircuit that is disjoint from both D_1 and D_2; a contradiction.

We now know that $x_1 \notin D_4$. Then, without loss of generality, $D_4 = \{x_2, x_6, x_9, x_{11}\}$. Similarly, if D_5 is a 4-cocircuit containing x_{12}, then D_5 is either $\{x_3, x_7, x_9, x_{12}\}$ or $\{x_3, x_7, x_{10}, x_{12}\}$. In the second case, $D_4 \cap D_5 = \emptyset$, so, by symmetry, this contradicts the assumption that D_3 and D_4 are not disjoint. We deduce that $D_5 = \{x_3, x_7, x_9, x_{12}\}$. Consider a 4-circuit C containing $\{x_4, x_9\}$. To avoid an orthogonality contradiction, C must meet each of $\{x_1, x_5, x_{10}\}$, $\{x_2, x_6, x_{11}\}$, and $\{x_3, x_7, x_{12}\}$. This contradiction completes the proof. $\square$

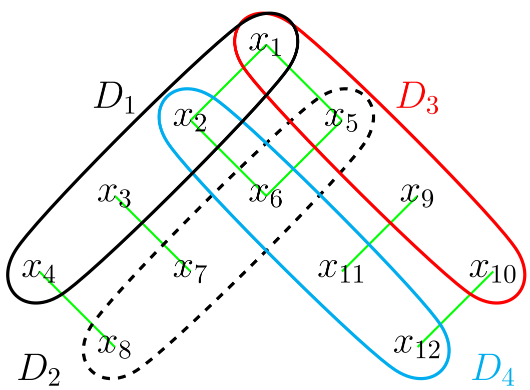

Figure 4.10: A set diagram of the structure of 4-cocircuits in Lemma 4.18.

The proof of the following proposition is similar to that of the preceeding lemma.

Proposition 4.19. *If $|E(M)| \geq 13$, then M has three pairwise-disjoint 4-cocircuits.*

Proof. Assume first the M has no three pairwise-disjoint 4-cocircuits. Let $E(M) = \{x_1, x_2, \ldots, x_n\}$. We know M has two disjoint 4-cocircuits, D_1 and D_2 by Proposition 4.16. Moreover, $M|(D_1 \cup D_2) \cong M(K_{2,4})$ by Proposition 4.6. Without loss of generality, let $D_1 = \{x_1, x_2, x_3, x_4\}$, $D_2 = \{x_5, x_6, x_7, x_8\}$, and $\{x_1, x_5\}$, $\{x_2, x_6\}$, $\{x_3, x_7\}$, and $\{x_4, x_8\}$

be the series pairs in $M|(D_1 \cup D_2)$. The elements in these pairs are always found together in circuits contained in $D_1 \cup D_2$. By assumption, x_9 is in a 4-cocircuit, say D_3. As D_3 meets $D_1 \cup D_2$, then by orthogonality with the 4-circuits contained in $M|(D_1 \cup D_2)$, we see that $D_3 \cap (D_1 \cup D_2)$ must be one of the aforementioned pairs. Without loss of generality, $D_3 = \{x_1, x_5, x_9, x_{10}\}$. Let D_4 be a 4-cocircuit containing x_{11}.

Claim 4.19.1. *$D_3 \cap D_4 \neq \emptyset$.*

Suppose $D_3 \cap D_4 = \emptyset$. Then, without loss of generality, $D_4 = \{x_2, x_6, x_{11}, x_{12}\}$. Now $M|(D_3 \cup D_4) \cong M(K_{2,4})$. The pairs $\{x_1, x_2\}$ and $\{x_5, x_6\}$ always appear together in the circuits contained in $D_3 \cup D_4$. Without loss of generality, so do $\{x_9, x_{11}\}$ and $\{x_{10}, x_{12}\}$. This is the configuration depicted in Figure 4.10..

A 4-cocircuit D_5 containing x_{13} must meet both $D_1 \cup D_2$ and $D_3 \cup D_4$. By orthogonality, the 4-circuits contained in $M|(D_1 \cup D_2)$ and $M|(D_3 \cup D_4)$ imply that D_5 meets $D_1 \cup D_2$ in $\{x_1, x_5\}$, $\{x_2, x_6\}$, $\{x_3, x_7\}$, or $\{x_4, x_8\}$, and D_5 meets $D_3 \cup D_4$ in $\{x_1, x_2\}$, $\{x_5, x_6\}$, $\{x_9, x_{11}\}$, or $\{x_{10}, x_{12}\}$. In each case, the first two possibilities cannot arise, otherwise M has $\{x_1, x_2, x_5, x_6\}$ as both a circuit and a cocircuit, a contradiction to the 4-connectedness of M. Hence $|D_5| \geq 5$, a contradiction. Thus Claim 4.19.1 holds.

We may assume, then, that D_3 meets D_4 and any further 4-cocircuits of M. Suppose $x_1 \in D_4$. Then we may assume that $D_4 = \{x_1, x_5, x_{11}, x_{12}\}$. Now $(D_3 \cup D_4) - x_1$ contains a cocircuit and, by orthogonality, this cocircuit avoids x_5. Hence $\{x_9, x_{10}, x_{11}, x_{12}\}$ is a cocircuit that is disjoint from both D_1 and D_2; a contradiction.

We now know that $x_1 \notin D_4$. Then, without loss of generality, $D_4 = \{x_2, x_6, x_9, x_{11}\}$. Similarly, if D_5 is a 4-cocircuit containing x_{12}, then D_5 is either $\{x_3, x_7, x_9, x_{12}\}$ or $\{x_3, x_7, x_{10}, x_{12}\}$. In the second case, $D_4 \cap D_5 = \emptyset$, so, by symmetry, we have a contradiction to Claim 4.19.1. We deduce that $D_5 = \{x_3, x_7, x_9, x_{12}\}$. Consider a 4-circuit C containing $\{x_4, x_9\}$. To avoid an orthogonality contradiction, C must meet each of $\{x_1, x_5, x_{10}\}$,

$\{x_2, x_6, x_{11}\}$, and $\{x_3, x_7, x_{12}\}$. This is impossible, as $|C| = 4$. This contradiction completes the proof. $\square$

4.3 When M Has Exactly Eight Elements

Throughout this section, we assume that $|E(M)| = 8$. The bulk of the examples on at most ten elements come from this case; as such, the analysis here is somewhat tedious. While we proceed in a more traditional manner, we concede that these results may possibly be checked by exhaustive computer search. In order to facilitate our analysis, we restrict M as follows. Observe that M must not have two disjoint 4-cocircuits, as Lemma 4.6 implies such an M must be isomorphic to $M(K_{2,4})$, a matroid that is not 4-connected. First, we show that $r(M) = 4$. Then, after proving a quick technical lemma, we treat the case when every 4-circuit of M meets every other in a single element. Finally, we address the remaining matroids in order of the maximum number of 4-circuits that may contain a particular element.

Lemma 4.20. *If $|E(M)| = 8$, then $r(M) = 4$.*

Proof. By $(P2)$, we know that M has a 4-cocircuit. That cocircuit must be independent, otherwise M is not 4-connected. Similarly, the 4-circuits of M must be coindependent. Since $r(M) + r^*(M) = |E(M)|$, it follows that $r(M) = 4$. $\square$

A consequence of the previous lemma is that the complement of every 4-circuit is a 4-cocircuit, and vice versa. Further, when coupled with the following result of Hartmanis [**?**], this observation guarantees that the objects we find in the main propositions of this section are actually matroids. Let k and m be integers with $k > 1$ and $m > 0$. Given a set E, we call a set $\mathcal{T} = \{T_1, T_2, \ldots, T_k\}$ an *m-partition* of E if each T_i is a subset of E with at least m elements, and each m-element subset of E is contained in a unique member of $\mathcal{T}$.

Proposition 4.21. *If $\mathcal{T}$ is an m-partition $\{T_1, T_2, \ldots, T_k\}$ of a set E, then $\mathcal{T}$ is the set of hyperplanes of a paving matroid of rank $m + 1$ on E.*

The following lemma, while easy, is quite useful. It states two obvious restrictions to the structure of 4-circuits of M. An important consequence of this lemma is that, whenever two 4-circuits C_1 and C_2 meet in a single element, every other 4-circuit containing that element must also contain the one element not in $C_1 \cup C_2$. This will frequently be referred to as the forced inclusion of an element; and, a sequence of forced inclusions will be called a *chain*.

Lemma 4.22. *Let C_1, C_2, and C_3 be distinct 4-circuits of M, and suppose $C_1 \cap C_2 = \{e\}$. Then $|C_3 \cap C_i| = 2$ for some $i \in \{1, 2\}$. Further, if $e \in C_3$, then $|C_3 \cap C_i| = 2$ for both $i \in \{1, 2\}$.*

Proof. For the first part, note that $|C_3 \cap (C_1 \cup C_2)| = 3$ since $|C_1 \cup C_2| = 7$. Hence the assertion follows by Proposition 4.2. That proposition also yields the second part. □

Corollary 4.23. *Let C_1, C_2, and C_3 be distinct 4-circuits of M. If $C_1 \cap C_2 = \{e\}$ and $e \in C_3$, then $E(M) - (C_1 \cup C_2) = \{f\} \in C_3$.*

We split the work of the main theorem of this section into two propositions. In the first proposition that follows, we examine matroids in which every 4-circuit meets another 4-circuit in a single element. We handle the remaining cases in the subsequent proposition. The list of 4-circuits of the matroids in these propositions are compiled in Figures 4.11 and 4.16 for reference.

Proposition 4.24. *If, for every 4-circuit C of M, there is a 4-circuit C' such that $|C \cap C'| = 1$, then M is one of the following matroids: $M_{8,1}$, $M_{8,2}$, $M_{8,3}$, $M_{8,3^+}$, $M_{8,4}$, $M_{8,4^+}$, $M_{8,5}$, $M_{8,6}$.*

Proof. Let $E(M) = \{x_1, x_2, \ldots, x_8\}$, and suppose C_1 is a 4-circuit of M. Without loss of generality, $C_1 = \{x_1, x_2, x_3, x_4\}$. By assumption, C_1 meets another 4-circuit in a single element. Let C_2 be such a 4-circuit; then, without loss of generality, $C_2 = \{x_1, x_5, x_6, x_7\}$. There is a 4-circuit, C_3, containing $\{x_1, x_8\}$. By Lemma 4.22, we may assume $C_3 = \{x_1, x_2, x_5, x_8\}$. We will refer to such a configuration of three 4-circuits as a *two-flap* configuration. The

M	The 4-circuits of M
$M_{8,1}$	$\{x_1, x_2, x_3, x_4\}$, $\{x_1, x_5, x_6, x_7\}$, $\{x_1, x_2, x_5, x_8\}$, $\{x_3, x_4, x_5, x_8\}$, $\{x_2, x_3, x_6, x_7\}$, $\{x_2, x_4, x_5, x_6\}$, $\{x_1, x_4, x_7, x_8\}$, $\{x_1, x_3, x_6, x_8\}$
$M_{8,2}$	$\{x_1, x_2, x_3, x_4\}$, $\{x_1, x_5, x_6, x_7\}$, $\{x_1, x_2, x_5, x_8\}$, $\{x_3, x_4, x_5, x_8\}$, $\{x_2, x_3, x_6, x_7\}$, $\{x_2, x_4, x_5, x_6\}$, $\{x_1, x_3, x_6, x_8\}$, $\{x_4, x_6, x_7, x_8\}$
$M_{8,3}$	$\{x_1, x_2, x_3, x_4\}$, $\{x_1, x_5, x_6, x_7\}$, $\{x_1, x_2, x_5, x_8\}$, $\{x_3, x_4, x_5, x_8\}$, $\{x_2, x_3, x_6, x_7\}$, $\{x_1, x_4, x_6, x_8\}$, $\{x_2, x_4, x_7, x_8\}$
$M_{8,3^+}$	$\{x_1, x_2, x_3, x_4\}$, $\{x_1, x_5, x_6, x_7\}$, $\{x_1, x_2, x_5, x_8\}$, $\{x_3, x_4, x_5, x_8\}$, $\{x_2, x_3, x_6, x_7\}$, $\{x_1, x_4, x_6, x_8\}$, $\{x_2, x_4, x_7, x_8\}$, $\{x_1, x_3, x_7, x_8\}$
$M_{8,4}$	$\{x_1, x_2, x_3, x_4\}$, $\{x_1, x_5, x_6, x_7\}$, $\{x_1, x_2, x_5, x_8\}$, $\{x_3, x_4, x_5, x_8\}$, $\{x_2, x_3, x_6, x_7\}$, $\{x_4, x_6, x_7, x_8\}$
$M_{8,4^+}$	$\{x_1, x_2, x_3, x_4\}$, $\{x_1, x_5, x_6, x_7\}$, $\{x_1, x_2, x_5, x_8\}$, $\{x_3, x_4, x_5, x_8\}$, $\{x_2, x_3, x_6, x_7\}$, $\{x_4, x_6, x_7, x_8\}$, $\{x_1, x_3, x_6, x_8\}$
$M_{8,5}$	$\{x_1, x_2, x_3, x_4\}$, $\{x_1, x_5, x_6, x_7\}$, $\{x_1, x_2, x_5, x_8\}$, $\{x_2, x_3, x_6, x_7\}$, $\{x_3, x_4, x_5, x_6\}$, $\{x_1, x_4, x_7, x_8\}$, $\{x_3, x_5, x_7, x_8\}$, $\{x_2, x_4, x_6, x_8\}$
$M_{8,6}$	$\{x_1, x_2, x_3, x_4\}$, $\{x_1, x_5, x_6, x_7\}$, $\{x_1, x_2, x_5, x_8\}$, $\{x_2, x_3, x_6, x_7\}$, $\{x_3, x_4, x_5, x_6\}$, $\{x_2, x_4, x_7, x_8\}$, $\{x_1, x_3, x_6, x_8\}$

Figure 4.11: The 4-circuits of the matroids in Proposition 4.24.

two-flap configuration here is depicted in Figure 4.12. By circuit elimination, we get four possible circuits $C_4 \subseteq (C_1 \cup C_3) - \{x_1\}$, $C_5 \subseteq (C_1 \cup C_3) - \{x_2\}$, $C_6 \subseteq (C_2 \cup C_3) - \{x_1\}$, and $C_7 \subseteq (C_2 \cup C_3) - \{x_5\}$.

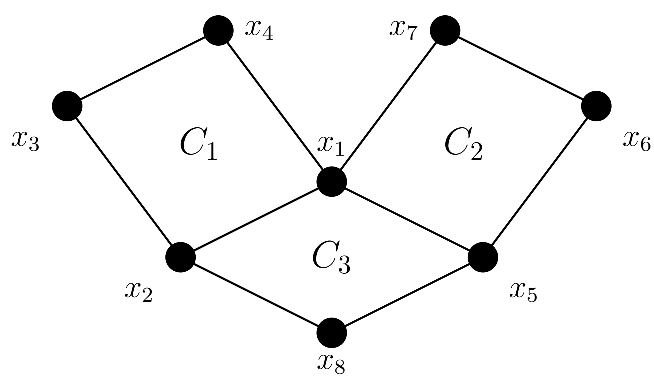

Figure 4.12: The two-flap configuration.

Claim 4.24.1. *There is at most one 4-circuit produced by circuit elimination on 4-circuits in a two-flap configuration.*

Suppose not. Then $C_4 = C_5 = \{x_3, x_4, x_5, x_8\}$, and $C_6 = C_7 = \{x_2, x_6, x_7, x_8\}$. Observe that $C_1 \cap C_2 = \{x_1\}$, $E(M) - (C_1 \cup C_2) = \{x_8\}$, $C_4 \cap C_6 = \{x_8\}$, and $E(M) - (C_4 \cup C_6) = \{x_1\}$. By Corollary 4.23, any further 4-circuit containing x_1 must contain x_8, and vice versa. Similarly, since $C_1 \cap C_6 = \{x_2\}$, $E(M) - (C_1 \cup C_6) = \{x_5\}$, $C_2 \cap C_4 = \{x_5\}$, and $E(M) - (C_2 \cup C_4) = \{x_2\}$, it follows by Corollary 4.23, that any further 4-circuit containing x_2 must contain x_5, and vice versa. But, by assumption, there must be a 4-circuit meeting C_3 in one element. This contradiction proves the claim.

Now, whenever three 4-circuits meet in a two-flap configuration, we know that circuit elimination on the pairs sharing two elements produces at most one 4-circuit. We will investigate these cases separately.

Case 4.24.2. *Suppose there exists a two-flap configuration that produces an additional 4-circuit via circuit elimination.*

We may assume C_1, C_2, and C_3 produce one such 4-circuit, say $C_4 = \{x_3, x_4, x_5, x_8\}$. As before, Corollary 4.23 applied to pairs $\{C_1, C_2\}$ and $\{C_2, C_4\}$ implies that further 4-circuits containing x_1 must contain x_8, and those containing x_5 must also contain x_2. Observe the symmetry between x_2 and x_8 and between x_3 and x_4, evident in Figure 4.13. We know

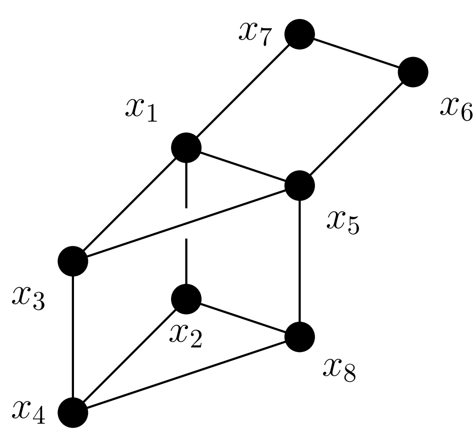

Figure 4.13: The 4-circuits in Case 4.24.2.

there must be a 4-circuit, C_5, meeting C_3 in a single element. The forced inclusions noted above indicate that C_5 must meet C_3 in either x_2 or x_8. Observe that the permutation $(x_1, x_5)(x_2, x_8)$ is an automorphism of M, and so we may assume $x_2 \in C_5$. The remaining elements of C_5 come from $E(M) - C_3 = \{x_3, x_4, x_6, x_7\}$. Since $E(M) - C_1 = \{x_5, x_6, x_7, x_8\}$ is a cocircuit, if C_5 contains one of x_6 or x_7, then it must contain the other to avoid an orthogonality contradiction. Therefore, $C_5 = \{x_2, x_3, x_6, x_7\}$, without loss of generality. Note that C_5 meets each of C_3 and C_4 in a single element. Corollary 4.23 applied to these pairs forces further 4-circuits containing x_2 to contain x_4 and those containing x_3 to contain x_1.

There must be a 4-circuit C_6 containing x_4 and x_6. We take two cases.

Subcase 4.24.2.1. *Assume C_6 does not contain x_8.*

Then $x_5 \in C_6$, otherwise $|C_6 \cap (E(M) - C_1)| = 1$, a contradiction by orthogonality. Further, $x_2 \in C_6$, otherwise it is orthogonal with one of $E(M) - C_2$ or $E(M) - C4$. Therefore, $C_6 = \{x_2, x_4, x_5, x_6\}$.

There must be a 4-circuit C_7 meeting C_6 in one element. By the forced inclusions, such a 4-circuit must contain either x_4 or x_6. We treat these cases separately, and each case will yield one matroid. We know that these, and all further examples, are matroids because the 4-circuits together with every 3-set that is in no 4-circuit form a 3-partition of $E(M)$, since no two such sets meet in more than two elements. This means that these sets are the hyperplanes of a paving matroid on $E(M)$, by Proposition 4.21.

Subcase 4.24.2.1.1. *Suppose $x_4 \in C_7$.*

As $(C_7 - \{x_4\}) \subseteq (E(M) - C_6) = \{x_1, x_3, x_7, x_8\}$, and, by forced inclusions, if $x_3 \in C_7$, then so must x_1 and x_8 be, we get that C_7 is either $\{x_1, x_3, x_4, x_8\}$ or $\{x_1, x_4, x_7, x_8\}$. However, the former set violates orthogonality with $E(M) - C_1$, so $C_7 = \{x_1, x_4, x_7, x_8\}$. With this, we get a long chain of forced inclusion of elements: if a further 4-circuit contains x_7, then it must contain x_5, and in turn must contain x_2, then x_4, then x_3, then x_1, and finally x_8. Thus, the only possible additional 4-circuits are $\{x_1, x_3, x_4, x_8\}$ or $\{x_1, x_3, x_6, x_8\}$. Further, both of these cannot be 4-circuits by Lemma 4.2. In this case, as x_6 and x_8 do not yet appear together in a 4-circuit, there must be a 4-circuit $C_8 = \{x_1, x_3, x_6, x_8\}$. This being the final possible 4-circuit of M, we conclude this subcase having determined our first matroid, which we call $M_{8,1}$.

Subcase 4.24.2.1.2. *We may now assume $x_4 \notin C_7$.*

Therefore C_7 contains x_6. As before, $(C_7 - \{x_6\}) \subseteq (E(M) - C_6) = \{x_1, x_3, x_7, x_8\}$. If $x_7 \in C_7$, then $x_3 \notin C_7$ because otherwise $|(E(M) - C_5) \cap C_7| = 1$. That implies $C_7 = \{x_1, x_6, x_7, x_8\}$, a contradiction since now $|(E(M) - C_2) \cap C_7| = 1$. Therefore, $x_7 \notin C_7$, and so $C_7 = \{x_1, x_3, x_6, x_8\}$. In this case, both pairs $\{x_4, x_7\}$ and $\{x_7, x_8\}$ are not in an identified 4-circuit. Consider a 4-circuit C_8 containing $\{x_4, x_7\}$. It must be that $x_8 \in C_8$, otherwise C_8 cannot contain two elements from each of the following cocircuits: $E(M) - C_1 = \{x_5, x_6, x_7, x_8\}$, $E(M) - C_2 = \{x_2, x_3, x_4, x_8\}$, $E(M) - C_5 = \{x_1, x_4, x_5, x_8\}$, and $E(M) - C_6 = \{x_1, x_3, x_7, x_8\}$. Then, we have x_3 and x_5 not in C_8, by applying Lemma 4.2 to C_8 and C_4. Therefore, C_8 is one of $\{x_1, x_4, x_7, x_8\}$, $\{x_2, x_4, x_7, x_8\}$, or $\{x_4, x_6, x_7, x_8\}$. If C_8 is either the first or second set, the resulting matroid is isomorphic to $M_{8,1}$; in the first case identically, and in the second case via the automorphism of M given by the permutation $(x_1, x_2)(x_3, x_5)(x_4, x_8)$. Therefore, we may assume $C_8 = \{x_4, x_6, x_7, x_8\}$. The forced inclusions determined by Corollary 4.23 are: containing x_1 forces x_8 which forces x_3 which forces x_1, and containing x_2 forces x_4 which forces x_5 which forces x_2, and finally x_6 forces the inclusion of x_7. Using this, we construct a short list of possible additional 4-circuits, all of which contradict orthogonality with some 4-cocircuit. Thus we have found a single matroid, which we call $M_{8,2}$.

Now case 4.24.2.1 is closed, and we may assume that $\{x_2, x_4, x_5, x_6\}$ is not a circuit. We return to C_6, which must now include x_8. The remaining element of C_6 must come from $(E(M) - C_4) - \{x_6\} = \{x_1, x_2, x_7\}$, in order to avoid an orthogonality contradiction.

However, x_1 and x_2 are symmetric under the automorphism given by $(x_1, x_2)(x_3, x_5)(x_4, x_8)$. Therefore, C_6 is either $\{x_1, x_4, x_6, x_8\}$ or $\{x_4, x_6, x_7, x_8\}$, without loss of generality.

Subcase 4.24.2.2. *Suppose* $C_6 = \{x_1, x_4, x_6, x_8\}$.

In this case, there is not yet a 4-circuit containing x_4 and x_7. Consider such a 4-circuit, and call it C_7. It is useful to consider the forced inclusions dictated by Corollary 4.23. Using this, we know that if C_7 contains x_3, it must also contain x_1, and consequently x_8; therefore, $x_3 \notin C_7$. Additionally, if $x_1 \in C_7$, then $C_7 = \{x_1, x_4, x_7, x_8\}$, and $|C_7 \cap C_8| = 3$, contradicting Lemma 4.2; therefore, $x_1 \notin C_7$. Similarly, if C_7 contains x_6, it must contain x_5 and then x_2, and then x_4, so $x_6 \notin C_7$. Further, if we suppose that $x_5 \in C_7$, then $C_7 = \{x_2, x_4, x_5, x_7\}$, which yields a matroid isomorphic to that considered in case 4.24.2.1, under the automorphism of M given by the permutation (x_6, x_7). Therefore, we may assume $x_5 \notin C_7$.

Thus $C_7 = \{x_2, x_4, x_7, x_8\}$. This extends the chains of forced inclusions. Since $|C_7 \cap C_2| = 1$, we get that a further 4-circuit containing x_7 must contain x_3. Therefore, the only possible additional 4-circuits allowed by the chains are $\{x_1, x_3, x_7, x_8\}$, $\{x_1, x_3, x_4, x_8\}$, $\{x_1, x_2, x_4, x_8\}$, $\{x_2, x_4, x_5, x_8\}$, and $\{x_2, x_4, x_5, x_6\}$. Each of these has a prohibitive intersection of size one with some 4-circuit, save $\{x_1, x_3, x_7, x_8\}$ and $\{x_2, x_4, x_5, x_6\}$. The latter set is out by assumption. Thus our analysis in this case produces two matroids: $M_{8,3}$ having 4-circuits $\{C_1, C_2, \ldots, C_7\}$, and $M_{8,3^+}$ having 4-circuits $\{C_1, C_2, \ldots, C_7\} \cup \{x_1, x_3, x_7, x_8\}$.

Subcase 4.24.2.3. *Suppose* $C_6 = \{x_4, x_6, x_7, x_8\}$.

There are no pairs of elements not in a 4-circuit. Therefore, we get a matroid $M_{8,4}$ having 4-circuits $\{C_1, C_2, \ldots, C_6\}$. This structure may admit additional 4-circuits, but such are subject to the following forced inclusions determined by Corollary 4.23: containing x_1 implies x_8, which implies x_3, which implies x_1, and containing x_2 implies x_4, which implies x_5, which implies x_2. Hence, any additional 4-circuits must be one of $\{x_1, x_3, x_6, x_8\}$, $\{x_1, x_3, x_7, x_8\}$, $\{x_2, x_4, x_5, x_6\}$, or $\{x_2, x_4, x_5, x_7\}$. The inclusion of any one of these sets as a 4-circuit produces an isomorphic matroid, as x_6 and x_7 are clones in $M_{8,4}$, and the permutation $(x_1, x_2)(x_3, x_5)(x_4, x_8)$ gives rise to an automorphism of M. Thus we get a second matroid, $M_{8,4^+}$ which has 4-circuits $\{C_1, C_2, \ldots, C_6\} \cup \{x_1, x_3, x_6, x_8\}$. This exhausts the case in which we assume a 4-circuit from the circuit elimination on C_1 and C_3.

Case 4.24.3. *Circuit elimination on 4-circuits in a two-flaps configuration produces no additional 4-circuits.*

As before, there is a 4-circuit, say C_4, meeting C_3 in a single element. That element cannot be x_1 by Corollary 4.23, as $\{x_1\} = C_1 \cap C_2$. Also, elements x_2 and x_5 are symmetric under the automorphism of M given by the permutation $(x_2, x_5)(x_3, x_6)(x_4, x_7)$. Therefore, it suffices to assume $C_3 \cap C_4$ is either $\{x_2\}$ or $\{x_8\}$.

Subcase 4.24.3.1. *Suppose $C_3 \cap C_4 = \{x_2\}$.*

Then $C_4 - \{x_2\} \subseteq (E(M) - C_3) = \{x_3, x_4, x_6, x_7\}$. Since $E(M) - C_1 = \{x_5, x_6, x_7, x_8\}$ is a cocircuit, if one of x_6 and x_7 is in C_4, then they both are. Therefore, $\{x_6, x_7\} \subseteq C_4$. Further, x_3 and x_4 are indistinguishable as they only appear thus far in the same 4-circuits, so without loss of generality we may assume $C_4 = \{x_2, x_3, x_6, x_7\}$.

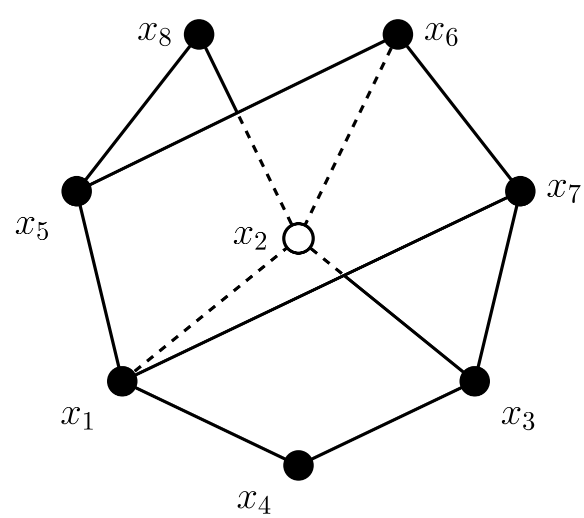

Figure 4.14: The configuration of 4-circuits in Case 4.24.3.1.

Now consider a 4-circuit C_5 containing $\{x_3, x_5\}$. Note that x_4 and x_8 are symmetric by the automorphism of M given by the permutation (x_1, x_2) $(x_3, x_5)(x_4, x_8)$. If $x_4 \notin C_5$, then, in order to avoid an orthogonality contradiction with $E(M) - C_4$, one of x_1 or x_8 must be in C_5. But, if $x_1 \in C_5$, then x_8 must also be in C_5 by Corollary 4.23. Therefore, in either case, $x_8 \in C_5$.

Therefore, without loss of generality, $x_4 \in C_5$. Hence, so must be one of x_6, x_7, or x_8 to avoid an orthogonality contradiction with $E(M) - C_1$. If $x_8 \in C_5$, then $C_5 = \{x_3, x_4, x_5, x_8\}$, which is a contradiction, as $C_5 \subseteq (C_1 \cup C_3) - \{x_1\}$. Therefore, as x_6 and x_7 are indistinguishable, we may assume $C_5 = \{x_3, x_4, x_5, x_6\}$.

Next, we consider a 4-circuit C_6 containing $\{x_4, x_7\}$. If $x_8 \notin C_6$, then, in order to avoid an orthogonality contradiction, C_6 must contain both x_2 and x_5; therefore, $C_6 = \{x_2, x_4, x_5, x_7\}$. There must be some 4-circuit C_7 meeting C_6 in a single element, by assumption. This element cannot be either x_2 or x_5, as these force the inclusion of x_4 and x_7, respectively. Observe that the permutation $(x_2, x_5)(x_3, x_6)(x_4, x_7)$ is an automorphism of M, so, without loss of generality, $x_4 \in C_7$. The rest of the elements of C_7 come from $E(M) - C_6 = \{x_1, x_3, x_6, x_8\}$. If $x_1 \notin C_7$, then $|C_7 \cap (E(M) - C_5)| = 1$, a contradiction. Therefore $x_1 \in C_7$. By Corollary 4.23, this forces $x_8 \in C_7$. Then, in order to avoid an orthogonality contradiction with $E(M) - C_1$, it must be that $x_6 \in C_7$. Therefore, $C_7 = \{x_1, x_4, x_6, x_8\}$. However, now $C_4 \cap C_7 = \{x_6\}$, and $C_6 \subseteq (C_4 \cup C_5) - \{x_6\}$, a contradiction. Thus $x_8 \in C_6$.

By the most recently cited automorphism, we may assume C_6 to be one of the following three sets, without loss of generality: $\{x_1, x_4, x_7, x_8\}$, $\{x_2, x_4, x_7, x_8\}$, or $\{x_3, x_4, x_7, x_8\}$.

Subcase 4.24.3.1.1. *Suppose* $C_6 = \{x_1, x_4, x_7, x_8\}$.

Now, since $C_4 \cap C_6 = \{x_7\}$ and $C_5 \cap C_6 = \{x_4\}$, further 4-circuits meeting $\{x_5, x_7\}$ or $\{x_2, x_4\}$ must contain both elements of that subset. Consider a 4-circuit C_7 containing x_3 and x_8. If $x_7 \notin C_7$, then neither is x_5. In order to avoid an orthogonality contradiction with $E(M) - C_1$, it must be that $x_6 \in C_7$. Now, neither x_2 nor x_4 may be elements of C_7, and so

$C_7 = \{x_1, x_3, x_6, x_8\}$. There must be a 4-circuit, C_8, that meets C_7 in a single element. They cannot share x_1, as that forces x_8, and x_3 and x_6 are isomorphic under the automorphism noted previously. Therefore, without loss of generality, either x_3 or x_8 are in C_8, with its other elements coming from $E(M) - C_7 = \{x_2, x_4, x_5, x_7\}$. However, the forced inclusions prevent C_8 from having only three elements from $E(M) - C_7$, a contradiction. Therefore $x_7 \in C_7$, and $C_7 = \{x_3, x_5, x_7, x_8\}$. In this case, there remains an undetermined 4-circuit, C_8, containing $\{x_6, x_8\}$. Note that x_3 and x_6 are symmetric in circuits C_1, C_2, ..., C_6. Therefore, determining the elements of C_8 is comparable to determining C_7. Either $x_2 \in C_8$, or we get a 4-circuit that cannot meet any other in a single element. Therefore, $C_8 = \{x_2, x_4, x_6, x_8\}$. There are no possible additional 4-circuits, as these eight determine, by Corollary 4.23, that any further 4-circuits must contain elements together in the following pairs: $\{x_1, x_8\}$, $\{x_2, x_4\}$, $\{x_3, x_6\}$, and $\{x_5, x_7\}$. One may quickly check that any 4-circuit containing any two of these pairs violates orthogonality with some 4-cocircuit of M. Thus this case determines a unique matroid, having 4-circuits $\{C_1, C_2, \ldots, C_8\}$, which we label $M_{8,5}$.

Subcase 4.24.3.1.2. *Suppose* $C_6 = \{x_2, x_4, x_7, x_8\}$*..*

Here $C_2 \cap C_6 = \{x_7\}$ and $C_5 \cap C_6 = \{x_4\}$. This gives two chains: inclusion of x_2 implies x_4, which implies x_1, which implies x_8, and inclusion of x_5 implies x_7, which implies x_3. Consider, again, a 4-circuit C_7 containing x_3 and x_8. By the chains, $x_2 \notin C_7$, and if $x_4 \in C_7$, then $C_7 = \{x_1, x_3, x_4, x_8\}$ which meets $E(M) - C_1$ in a single element, a contradiction. Therefore $x_4 \notin C_7$.

Consider when $x_5 \in C_7$. The chains noted above indicate that having x_5 forces the inclusion of x_7, which forces x_3. Therefore $C_7 = \{x_3, x_5, x_7, x_8\}$. In this case, there must be a 4-circuit C_8 containing the pair $\{x_6, x_8\}$. Since $C_1 \cap C_7 = \{x_3\}$, further 4-circuits containing x_3 must contain x_6. In light of these chains, C_8 must be one of $\{x_3, x_6, x_7, x_8\}$, $\{x_1, x_4, x_6, x_8\}$, or $\{x_1, x_3, x_6, x_8\}$. The first of these is out by orthogonality with $E(M) - C_7$, and the third is out as $C_7 \subseteq (C_2 \cup \{x_1, x_3, x_6, x_8\}) - \{x_1\}$. Now suppose $C_8 = \{x_1, x_4, x_6, x_8\}$. In this case we get a matroid. This choice gives two further forced inclusions: $C_4 \cap C_8 = \{x_6\}$, so x_6 forces x_5, and $C_7 \cap C_8 = \{x_8\}$, so x_8 forces x_2. Therefore, the only possible additional 4-circuits are $\{x_1, x_2, x_4, x_8\}$ and $\{x_3, x_5, x_6, x_7\}$, each of these leading to an orthogonality contradiction. Thus we get one matroid, with 4-circuits $\{C_1, C_2, \ldots, C_8\}$. However, this matroid is isomorphic to $M_{8,5}$ via the automorphism given by the permutation $(x_1, x_2)(x_3, x_5)(x_4, x_8)(x_6, x_7)$.

We may assume, then, that $x_5 \notin C_7$. In this case, if $x_6 \notin C_7$, then $C_7 = \{x_1, x_3, x_7, x_8\}$, which is a contradiction as $C_6 \subseteq (C_1 \cup \{x_1, x_3, x_7, x_8\}) - \{x_1\}$. Therefore, $x_6 \in C_7$. Then, in order to avoid an orthogonality contradiction with either $E(M) - C_4 = \{x_1, x_4, x_5, x_8\}$ or $\{x_1, x_2, x_7, x_8\}$, it must be that $C_7 = \{x_1, x_3, x_6, x_8\}$. We get the forced inclusion x_8 implies x_5, so the only possible additional 4-circuits are $\{x_3, x_5, x_7, x_8\}$ and $\{x_3, x_5, x_6, x_7\}$. These both lead to contradictions: in the first case, $C_2 \subseteq (C_7 \cup \{x_3, x_5, x_7, x_8\}) - \{x_3\}$, and in the second, $|(E(M) - C_2) \cap \{x_3, x_5, x_6, x_7\}| = 1$. Thus we get one matroid, with 4-circuits $\{C_1, C_2, \ldots, C_7\}$, which we call $M_{8,6}$.

Subcase 4.24.3.1.3. *Suppose* $C_6 = \{x_3, x_4, x_7, x_8\}$.

In this case we get the following chains: x_5 implies x_7, which implies x_2, which implies x_4, and x_1 implies x_8, which implies x_6. With this in mind, consider a 4-circuit C_7 containing $\{x_6, x_8\}$. By the chains of forced inclusion, it is clear that neither x_5 nor x_7 may be in C_7. Additionally, if $x_1 \notin C_7$, then, to avoid an orthogonality contradiction with $E(M) - C_3$ or $E(M) - C_6$, it must be that $C_7 = \{x_2, x_4, x_6, x_8\}$. But then $C_4 \subseteq (C_6 \cup C_7) - \{x_8\}$, a contradiction. Therefore, $x_1 \in C_7$. The forced inclusions prove that $x_2 \notin C_7$, and so C_7 is one of $\{x_1, x_3, x_6, x_8\}$ or $\{x_1, x_4, x_6, x_8\}$. In the first case, there is no 4-circuit that meets $\{x_1, x_3, x_6, x_8\}$ in a single element. Such a 4-circuit would have to contain either x_3 or x_6, with the rest of its elements coming from $E(M) - \{x_1, x_3, x_6, x_8\} = \{x_2, x_4, x_5, x_7\}$. By the chains of forced inclusion, these 4-circuits would necessarily be $\{x_2, x_3, x_4, x_7\}$ and $\{x_2, x_4, x_6, x_7\}$, respectively, both of which are out by orthogonality with $E(M) - C_4$. Therefore, the only possibility remaining is that $C_7 = \{x_1, x_4, x_6, x_8\}$. Here we get a new forced inclusion: x_6 forces

x_5. This creates a long chain of forced inclusions, which dictate that any additional 4-circuit must be either $\{x_2, x_3, x_4, x_7\}$, or $\{x_2, x_4, x_5, x_7\}$. The first of these is out by orthogonality with $E(M) - C_1$. In the second case, $\{x_2, x_4, x_5, x_7\} \subseteq (C_4 \cup C_5) - \{x_6\}$, a contradiction. Thus, this case provides one matroid, with 4-circuits $\{C_1, C_2, \ldots, C_7\}$. However, this matroid is isomorphic to $M_{8,6}$, via the automorphism given by $(x_1, x_2, x_7, x_5, x_8, x_4, x_3, x_6)$.

Subcase 4.24.3.2. *Suppose* $C_3 \cap C_4 = \{x_8\}$.

Our initial assumptions in Case 4.24.3.1 produced the configuration in Figure 4.14. As we have now exhausted the possible matroids that arise from that configuration, we may from now on assume the configuration in Figure 4.14 is disallowed. We will refer such an arrangement of 4-circuits as the forbidden configuration. With only C_1, C_2, and C_3 determined, the elements x_3 and x_4 are indistinguishable, as are x_6 and x_7. As C_4 must contain a second element from each of $E(M) - C_1$ and $E(M) - C_2$ in order to avoid an orthogonality contradiction, we may assume, without loss of generality, that $\{x_3, x_6\} \subseteq C_4$. The fourth element of C_4 must be either x_4 or x_7, and these are symmetric choices, as the permutation $(x_2, x_5)(x_3, x_6)(x_4, x_7)$ is an automorphism of M. Therefore, it suffices to assume $C_4 = \{x_3, x_4, x_6, x_8\}$. Note that x_3 and x_4 remain symmetric. Now, there must be a 4-circuit C_5 containing x_2 and x_7.

Subcase 4.24.3.2.1. *Suppose* $x_8 \in C_5$.

To avoid an orthogonality contradiction with $E(M)-C_3$, we must have one of x_3, x_4, and x_6 in C_5. If $x_6 \in C_5$, then C_1, C_3, C_4, and C_5 form the forbidden configuration, a contradiction. Therefore, we may assume $C_5 = \{x_2, x_3, x_7, x_8\}$, without loss of generality. Note that the single-element intersections of the known 4-circuits give the following chains: inclusion of x_1 implies x_8, which implies x_7, which implies x_4, and the inclusion of x_6 implies x_2. Now, there must be an 4-circuit, C_6, containing $\{x_2, x_6\}$. By the forced inclusions, neither x_1 nor x_8 is in C_6. If $x_7 \in C_6$, then $C_6 = \{x_2, x_4, x_6, x_7\}$, which forms the forbidden configuration with C_2,

C_3, and C_5. Therefore, $x_7 \notin C_6$. To avoid an orthogonality contradiction with $E(M) - C_4$, it must be that $x_5 \in C_6$. Therefore, C_6 is either $\{x_2, x_4, x_5, x_6\}$ or $\{x_2, x_3, x_5, x_6\}$. Both cases lead to a contradiction. In the first case, C_3, C_4, C_5, and C_6 form the forbidden configuration. In the second case, there must be a 4-circuit, C_7, meeting C_6 in a single element. That element cannot be x_6 by forced inclusion of x_2. Therefore, C_7 contains one of x_2, x_3, or x_5, with its remaining elements coming from $E(M) - C_6 = \{x_1, x_4, x_7, x_8\}$. By the long chain of forced inclusions, this means C_7 is one of $\{x_2, x_4, x_7, x_8\}$, $\{x_3, x_4, x_7, x_8\}$, or $\{x_4, x_5, x_7, x_8\}$. The first two choices give contradictions to orthogonality with $E(M) - C_5$, and the final choice, together with C_2, C_5, and C_6 creates the forbidden configuration. Thus there are no viable matroids when $x_8 \in C_5$. We have now reduced to that following:

Subcase 4.24.3.2.2. $x_8 \notin C_5$.

Now, one element fom each of $\{x_3, x_4\}$ and $\{x_5, x_6\}$ must be in C_5, otherwise C_5 meets each of $E(M) - C_2$ and $E(M) - C_1$, respectively, in a single element. We may assume $x_3 \in C_5$ without loss of generality, and so C_5 is either $\{x_2, x_3, x_5, x_7\}$ or $\{x_2, x_3, x_6, x_7\}$. In the latter case, C_1, C_2, C_3, and C_5 form the forbidden configuration. Therefore, we need only consider the former case. This extends one chain of forced inclusions, with x_3 implying x_1, which implies x_8, which implies x_7. There must be a 4-circuit C_6 containing $\{x_2, x_6\}$. The noted chain gives $x_3 \notin C_6$ and $x_1 \notin C_6$. If $x_4 \notin C_6$, then $C_6 = \{x_2, x_6, x_7, x_8\}$, otherwise it violates orthogonality with one of $E(M) - C_2$ or $E(M) - C_3$. But, then C_1, C_3, C_4, and C_6 form the forbidden configuration. Therefore, $x_4 \in C_6$, and C_6 is either $\{x_2, x_4, x_5, x_6\}$ or $\{x_2, x_4, x_6, x_7\}$. In the latter case, C_1, C_2, C_3, and C_6 form the forbidden configuration, so assume $C_6 = \{x_2, x_4, x_5, x_6\}$. There must be a 4-circuit C_7 that meets C_6 in a single element, and that element cannot be x_6, as x_6 forces x_2 as well. By the long chain of forced inclusions in this case, C_7 must be one of $\{x_1, x_2, x_7, x_8\}$, $\{x_1, x_4, x_7, x_8\}$, or $\{x_1, x_5, x_7, x_8\}$. The first of these violates orthogonality with $E(M) - C_3$, and the third with $E(M) - C_2$. This leaves the possibility that $C_7 = \{x_1, x_4, x_7, x_8\}$, but then C_2, C_4, C_6, and C_7 form the forbidden configuration. Thus, there are no matroids possible in this case. This concludes the analysis when we assume every 4-circuit meets some other in a single element, and we have found eight matroids.

To see that these matroids are all unique, we perform the following analysis. First, we distinguish the matroids using a count of their 4-circuits. Next, we assign an 8-tuple $(w_1, w_2, \ldots, w_8)$ to each matroid, where w_i is the number of distinct 4-circuits of M containing x_i. This is sufficient to determine the uniqueness of each matroid, as summarized in Figure 4.15. $\square$

Matroid	4-Circuit #	Element Weights	Comment
$M_{8,1}$	8	$(5,4,4,4,4,4,3,4)$	The elements with weights 3 and 5 appear together twice in 4-circuits.
$M_{8,2}$	8	$(4,4,4,4,4,5,3,4)$	The elements with weights 3 and 5 appear together three times in 4-circuits.
$M_{8,3}$	7	$(4,4,3,4,3,3,3,4)$	Two elements with weight 4 appear in 4-circuits without any other elements of weight 4.
$M_{8,3^+}$	8	$(5,4,4,4,3,3,4,5)$	There are two element with weight 5.
$M_{8,4}$	6	$(3,3,3,3,3,3,3,3)$	This matroid only has six 4-circuits.
$M_{8,4^+}$	7	$(4,3,4,3,3,4,3,4)$	All elements with weight 4 appear together in a 4-circuit.
$M_{8,5}$	8	$(4,4,4,4,4,4,4,4)$	All elements have weight 4.
$M_{8,6}$	7	$(4,4,4,3,3,4,3,3)$	Only one element with weight 4 appears in a 4-circuit without any other elements of weight 4.

Figure 4.15: Evidence for the uniqueness of each matroid determined in Proposition 4.24.

From here on, we change our tack. We may now assume M has at least one 4-circuit that does not meet any other in a single element. Our strategy in the following proof will be to progressively limit the number of 4-circuits that may contain a specific element of M.

Proposition 4.25. *If M contains a 4-circuit C such that $|C \cap C'| \neq 1$ for all 4-circuits $C' \in \mathcal{C}(M)$, then M is one of the following matroids: $M_{8,7}$, $M_{8,7^+}$, $M_{8,8a}$, $M_{8,8b}$, $M_{8,9a}$, $M_{8,9b}$, $M_{8,9b^+}$, $M_{8,10}$, $M_{8,10^+}$, $M_{8,10^{++}}$, $M_{8,11}$, $M_{8,12}$, and F_7^+.*

M	The 4-circuits of M
$M_{8,7}$	$\{x_1,x_2,x_3,x_4\}$, $\{x_1,x_2,x_5,x_6\}$, $\{x_1,x_2,x_7,x_8\}$, $\{x_1,x_3,x_5,x_7\}$, $\{x_1,x_3,x_6,x_8\}$, $\{x_1,x_4,x_5,x_8\}$, $\{x_2,x_4,x_6,x_7\}$
$M_{8,7^+}$	$\{x_1,x_2,x_3,x_4\}$, $\{x_1,x_2,x_5,x_6\}$, $\{x_1,x_2,x_7,x_8\}$, $\{x_1,x_3,x_5,x_7\}$, $\{x_1,x_3,x_6,x_8\}$, $\{x_1,x_4,x_5,x_8\}$, $\{x_2,x_4,x_6,x_7\}$, $\{x_3,x_4,x_6,x_7\}$
$M_{8,8a}$	$\{x_1,x_2,x_3,x_4\}$, $\{x_1,x_2,x_5,x_6\}$, $\{x_1,x_2,x_7,x_8\}$, $\{x_1,x_3,x_5,x_7\}$, $\{x_1,x_3,x_6,x_8\}$, $\{x_2,x_4,x_5,x_8\}$, $\{x_3,x_4,x_6,x_7\}$
$M_{8,8b}$	$\{x_1,x_2,x_3,x_4\}$, $\{x_1,x_2,x_5,x_6\}$, $\{x_1,x_2,x_7,x_8\}$, $\{x_1,x_3,x_5,x_7\}$, $\{x_1,x_3,x_6,x_8\}$, $\{x_2,x_4,x_5,x_8\}$, $\{x_2,x_4,x_6,x_7\}$
$M_{8,9a}$	$\{x_1,x_2,x_3,x_4\}$, $\{x_1,x_2,x_5,x_6\}$, $\{x_1,x_2,x_7,x_8\}$, $\{x_1,x_3,x_5,x_7\}$, $\{x_1,x_4,x_5,x_8\}$, $\{x_2,x_3,x_6,x_8\}$, $\{x_2,x_4,x_6,x_7\}$
$M_{8,9b}$	$\{x_1,x_2,x_3,x_4\}$, $\{x_1,x_2,x_5,x_6\}$, $\{x_1,x_2,x_7,x_8\}$, $\{x_1,x_3,x_5,x_7\}$, $\{x_1,x_4,x_5,x_8\}$, $\{x_2,x_3,x_6,x_8\}$, $\{x_3,x_4,x_6,x_7\}$
$M_{8,9b^+}$	$\{x_1,x_2,x_3,x_4\}$, $\{x_1,x_2,x_5,x_6\}$, $\{x_1,x_2,x_7,x_8\}$, $\{x_1,x_3,x_5,x_7\}$, $\{x_1,x_4,x_5,x_8\}$, $\{x_2,x_3,x_6,x_8\}$, $\{x_3,x_4,x_6,x_7\}$, $\{x_2,x_4,x_5,x_7\}$
$M_{8,10}$	$\{x_1,x_2,x_3,x_4\}$, $\{x_1,x_2,x_5,x_6\}$, $\{x_1,x_2,x_7,x_8\}$, $\{x_1,x_3,x_5,x_7\}$, $\{x_1,x_4,x_6,x_8\}$, $\{x_3,x_4,x_5,x_8\}$, $\{x_3,x_4,x_6,x_7\}$
$M_{8,10^+}$	$\{x_1,x_2,x_3,x_4\}$, $\{x_1,x_2,x_5,x_6\}$, $\{x_1,x_2,x_7,x_8\}$, $\{x_1,x_3,x_5,x_7\}$, $\{x_1,x_4,x_6,x_8\}$, $\{x_3,x_4,x_5,x_8\}$, $\{x_3,x_4,x_6,x_7\}$, $\{x_2,x_3,x_6,x_8\}$
$M_{8,10^{++}}$	$\{x_1,x_2,x_3,x_4\}$, $\{x_1,x_2,x_5,x_6\}$, $\{x_1,x_2,x_7,x_8\}$, $\{x_1,x_3,x_5,x_7\}$, $\{x_1,x_4,x_6,x_8\}$, $\{x_3,x_4,x_5,x_8\}$, $\{x_3,x_4,x_6,x_7\}$, $\{x_2,x_3,x_6,x_8\}$, $\{x_2,x_4,x_5,x_7\}$
$M_{8,11}$	$\{x_1,x_2,x_3,x_4\}$, $\{x_1,x_2,x_5,x_6\}$, $\{x_1,x_2,x_7,x_8\}$, $\{x_1,x_3,x_5,x_7\}$, $\{x_1,x_4,x_6,x_8\}$, $\{x_3,x_4,x_5,x_8\}$, $\{x_2,x_3,x_6,x_7\}$, $\{x_2,x_4,x_5,x_7\}$
$M_{8,12}$	$\{x_1,x_2,x_3,x_4\}$, $\{x_1,x_2,x_5,x_6\}$, $\{x_1,x_3,x_5,x_7\}$, $\{x_1,x_4,x_5,x_8\}$, $\{x_2,x_3,x_7,x_8\}$, $\{x_2,x_4,x_6,x_7\}$, $\{x_3,x_4,x_6,x_8\}$

Figure 4.16: The 4-circuits of the matroids in Proposition 4.25.

Proof. Let C_* be a 4-circuit of M that meets no other 4-circuit in a single element, and let $C_* = \{x_1, x_2, x_3, x_4\}$. Note that this implies that every other 4-circuit of M meets C_* in exactly two elements, as $E(M) - C_*$ is a cocircuit. We may assume that, of all the elements in C_*, the element x_1 is contained in the most 4-circuits. Let C_1, C_2, ..., C_n be the list of 4-circuits distinct from C_* that contain x_1. For each i and j in $\{1, 2, \ldots, n\}$, if $C_i \cap C_* = C_j \cap C_*$, then either $i = j$, or $C_i \cap C_j \cap (E(M) - C_*) = \emptyset$. Further, as each C_i contains x_1, it must be that $C_i \cap (E(M) - C_*) \neq C_j \cap (E(M) - C_*)$ when $i \neq j$. We divide the work that follows into cases determined by the maximum value of n.

Case 4.25.1. *Suppose* $n \geq 6$.

We may assume $C_1 = \{x_1, x_2, x_5, x_6\}$ and $C_2 = \{x_1, x_2, x_7, x_8\}$, without loss of generality. No further 4-circuits may contain both x_1 and x_2 without meeting one of C_1, C_2, or C_3 in at least three elements, which violates orthogonality as the complements of 4-circuits are 4-cocircuits. Therefore, we may assume that $x_3 \in C_3$. Without loss of generality, $C_3 = \{x_1, x_3, x_5, x_7\}$, and so we may assume $C_4 = \{x_1, x_3, x_6, x_8\}$. The only possible additional 4-circuits containing x_1 must also contain x_4, so, as before and without loss of generality, we may assume $C_5 = \{x_1, x_4, x_5, x_8\}$ and $C_6 = \{x_1, x_4, x_6, x_7\}$. It is clear there can be no more 4-circuits containing x_1. Indeed, there can be no further 4-circuits, as every other 4-element set meeting C_* in two elements either shares three elements with one of C_1, C_2, ..., C_6, or is disjoint from them, violating orthogonality and 4-connectivity, respectively. The matroid in this case is recognizable as the unique free coextenstion of the Fano matroid. Thus, the maximum number of additional 4-circuits containing x_1 is six, and there is one example when $n \geq 6$.

Case 4.25.2. *Suppose* $n = 5$.

As in Case 4.25.1, we may assume, without loss of generality, that the pairs $\{x_1, x_2\}$ and $\{x_1, x_3\}$ appear twice in other 4-circuits, and the pair $\{x_1, x_4\}$ appears once. We are free, then, to assume that $C_1 = \{x_1, x_2, x_5, x_6\}$, $C_2 = \{x_1, x_2, x_7, x_8\}$, $C_3 = \{x_1, x_3, x_5, x_7\}$, $C_4 = \{x_1, x_3, x_6, x_8\}$, and $C_5 = \{x_1, x_4, x_5, x_8\}$, as before. Now, however, there is not yet a

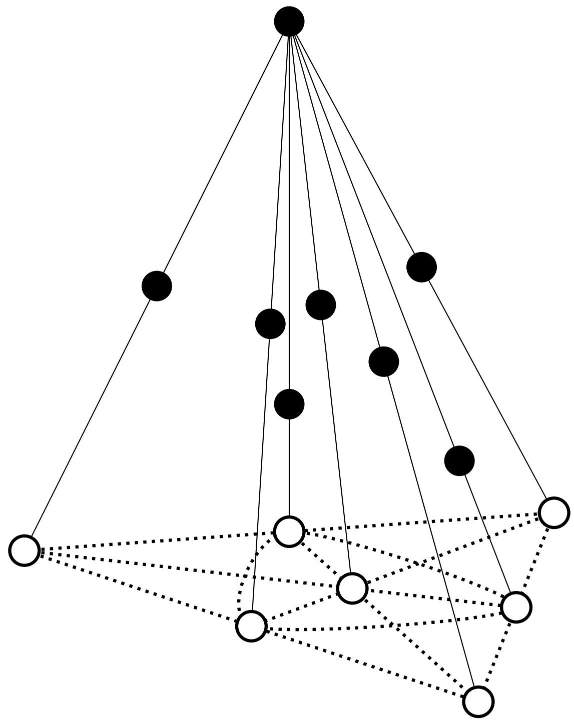

Figure 4.17: The matroid F_7^+.

4-circuit containing $\{x_4, x_6\}$. Let D_1 be such a 4-circuit. Without loss of generality, $x_2 \in D_1$, as x_2 and x_3 are symmetric via the automorphism given by the permutation $(x_2, x_3)(x_5, x_8)$. Proceeding, $x_5 \notin D_1$, otherwise $|D_1 \cap (E(M) - C_1)| = 1$, and $x_8 \notin D_1$, otherwise D_1 and C_3 are disjoint. This implies $D_1 = \{x_2, x_4, x_6, x_7\}$. We label this matroid $M_{8,7}$, having 4-circuits C_1, C_2, C_3, C_4, C_5, and D_1.

It is possible that this structure permits an additional 4-circuit. Such a circuit, call it D_2, must contain a pair from $\{x_2, x_3, x_4\}$ and from $\{x_5, x_6, x_7, x_8\}$. If $\{x_2, x_4\} \subseteq D_2$, then the only possible pairs from $\{x_5, x_6, x_7, x_8\}$ are those which do not already appear in a 4-circuit containing either x_2 or x_4. That leaves $\{x_5, x_7\}$ and $\{x_6, x_8\}$; however, $\{x_2, x_4, x_5, x_7\}$ and $\{x_2, x_4, x_6, x_8\}$ are disjoint from $E(M) - C_4$ and $E(M) - C_3$, respectively. This is a contradiction, and so $\{x_2, x_4\} \not\subseteq D_2$. The case when $\{x_3, x_4\} \subseteq D_2$ is similar. Here, the only viable pairs from $\{x_5, x_6, x_7, x_8\}$ are $\{x_5, x_6\}$ and $\{x_7, x_8\}$, each of which leads to a connectivity contradiction, as before. Therefore $\{x_3, x_4\} \not\subseteq D_2$. The last case has $\{x_2, x_3\} \subseteq D_2$. Again, the possible remaining elements of D_2 are either $\{x_5, x_8\}$ or $\{x_6, x_7\}$. The latter choice gives $D_2 = \{x_2, x_3, x_6, x_7\}$ which is disjoint from C_5, a contradiction. However, there is no problem with $D_2 = \{x_2, x_3, x_5, x_8\}$. Thus we have a second matroid in this case, which we call $M_{8,7^+}$, and which has 4-circuits C_1, C_2, C_3, C_4, C_5, D_1, and D_2.

Case 4.25.3. *Suppose $n = 4$.*

We may now assume that an element of C_* is contained in at most four other 4-circuits. These circuits meet C_* in one of $\{x_1, x_2\}$, $\{x_1, x_3\}$, or $\{x_1, x_4\}$. There are two non-isomorphic ways this may happen: either two pairs are used twice, and one pair not at all; or one pair is used twice, and the others are used once. We treat these in cases.

Subcase 4.25.3.1. *Assume two pairs are used twice, and one pair not at all.*

We may assume the pairs in question are $\{x_1, x_2\}$ and $\{x_1, x_3\}$. Without loss of generality, we get $C_1 = \{x_1, x_2, x_5, x_6\}$, $C_2 = \{x_1, x_2, x_7, x_8\}$, $C_3 = \{x_1, x_3, x_5, x_7\}$, and $C_4 = \{x_1, x_3, x_6, x_8\}$. There must be a 4-circuit, say D_1 containing x_4 and x_5. As D_1 must meet C_* in two elements, and as x_1 cannot be in D_1 by assumption, we may assume without loss of generality that $x_2 \in D_1$. Now, $x_6 \notin D_1$ otherwise we obtain an orthogonality contradiction with $E(M) - C_1$; and $x_7 \notin D_1$ since $\{x_2, x_4, x_5, x_7\} = E(M) - C_3$ is a cocircuit. Therefore, $D_1 = \{x_2, x_4, x_5, x_8\}$. There must also be a 4-circuit, D_2, containing $\{x_4, x_6\}$. From this we get two cases.

Subcase 4.25.3.1.1. *Suppose $x_3 \in D_2$.*

In order to avoid an orthogonality contradiction with $E(M) - C_4$, one of x_5 and x_7 must be in D_2; however, $\{x_3, x_4, x_5, x_6\} = E(M) - C_2$, so $x_7 \in D_2$. Now $D_2 = \{x_5, x_6, x_7, x_8\}$, and there are no further 4-circuits possible, as every pair from $\{x_5, x_6, x_7, x_8\}$ has appeared with one of x_2, x_3, or x_4 in one of the known 4-circuits. Therefore we get one matroid, which we denote $M_{8,8a}$.

Subcase 4.25.3.1.2. *Suppose $x_3 \notin D_2$.*

This forces $x_7 \in D_2$, otherwise $|D_2 \cap (E(M) - D_1)| = 1$, and so $D_2 = \{x_2, x_4, x_6, x_7\}$. Again, there are no further 4-circuits possible. An additional 4-circuit would necessarily include x_3 and x_4, and the only pair from $E(M) - C_*$ not appearing in a known 4-circuit with either of those elements is $\{x_5, x_6\}$. But $\{x_3, x_4, x_5, x_6\} = E(M) - C_2$, a contradiction. Thus we get a second matroid from these cases, which we call $M_{8,8b}$.

Subcase 4.25.3.2. *Assume one pair is used twice and the others are used once*

Without loss of generality, suppose both C_1 and C_2 contain $\{x_1, x_2\}$, while C_3 and C_4 contain $\{x_1, x_3\}$ and $\{x_1, x_4\}$, respectively. It suffices to assume $C_1 = \{x_1, x_2, x_5, x_6\}$ and $C_2 = \{x_1, x_2, x_7, x_8\}$. From here there are two possibilities: either $|C_3 \cap C_4| = 2$ or $|C_3 \cap C_4| = 1$.

Subcase 4.25.3.2.1. $|C_3 \cap C_4| = 2$.

We may assume $C_3 = \{x_1, x_3, x_5, x_7\}$ and $C_4 = \{x_1, x_4, x_5, x_8\}$. In this case, consider a 4-circuit, D_1, containing x_3 and x_6.

Suppose x_2 is in D_1. Then D_1 must contain one of x_7 and x_8 in order to avoid an orthogonality contradiction with $E(M) - C_1$. Since $\{x_2, x_3, x_6, x_7\} = E(M) - C_4$, we get $D_1 = \{x_2, x_3, x_6, x_8\}$. Consider a 4-circuit D_2 containing x_4 and x_6. Such a circuit must contain one of x_5 and x_7 to avoid an orthogonality contradiction with $E(M) - D_1$. However, $x_5 \notin D_2$, as then neither x_2 nor x_3 may be members of D_2 without violating orthogonality with some 4-cocircuit of M. Therefore, D_2 is one of $\{x_2, x_4, x_6, x_7\}$ and $\{x_3, x_4, x_6, x_7\}$. If we allow D_2 to be the former set, we get a matroid that admits no further 4-circuits, which we call $M_{8,9a}$. If we allow D_2 to be the latter set, we get a second matroid which we call $M_{8,9b}$. This, however, does admit one further 4-circuit. It is possible that $\{x_2, x_4, x_5, x_7\}$ is a circuit in addition to those of $M_{8,9b}$ without producing contradictions. We call this third matroid in this case $M_{8,9b^+}$.

Now assume that x_2 is not in D_1. Then $D_1 = \{x_3, x_4, x_6, x_7\}$ without loss of generality, as x_7 and x_8 are symmetric given only C_*, C_1, C_2, C_3, and C_4. Now, there must be a 4-circuit, D_2, containing $\{x_3, x_8\}$. In order to avoid an orthogonality contradiction with one of $E(M) - C_4$ or $E(M) - D_1$, such a circuit must contain x_2. Further, D_2 must also contain one of x_5 and x_6, otherwise it violates orthgonality with $E(M) - C_2$. But $\{x_2, x_3, x_6, x_8\}$ was considered in the previous case, so $D_2 = \{x_2, x_3, x_5, x_6\}$. Now, there yet must be a 4-circuit containing x_6 and x_8. Such a circuit cannot contain either $\{x_2, x_3\}$ or $\{x_3, x_4\}$, otherwise we get an orthogonality contradiction. Further, $\{x_2, x_4, x_6, x_8\} = E(M) - C_3$, and cannot be a circuit. Thus $M_{8,9a}$, $M_{8,9b}$, and $M_{8,9c}$ are the only matroids determined by this case.

Subcase 4.25.3.2.2. $|C_3 \cap C_4| = 1$.

We may assume $C_3 = \{x_1, x_3, x_5, x_7\}$ and $C_4 = \{x_1, x_4, x_6, x_8\}$. If x_3 and x_4 do not appear together in an additional 4-circuit, then, by the case restriction since x_2 already appears twice with x_1, there are at most two more 4-circuits in M, one containing $\{x_2, x_3\}$ and the other containing $\{x_2, x_4\}$. Now, there must be a 4-circuit containing $\{x_4, x_7\}$, and also a 4-circuit containing $\{x_6, x_7\}$, so it must be that $\{x_2, x_4, x_6, x_7\}$ is a circuit. But, there must also be a 4-circuit containing $\{x_4, x_5\}$, a contradiction.

Therefore there must be some 4-circuit, say D_1, containing $\{x_3, x_4\}$. We may assume that $x_5 \in D_1$, as any two elements from $\{x_5, x_6, x_7, x_8\}$ are symmetric, since both permutations $(x_3, x_4)(x_5, x_6)(x_7, x_8)$ and $(x_5, x_7)(x_6, x_8)$ are automorphisms of M. This implies that $D_1 = \{x_3, x_4, x_5, x_8\}$, as D_1 having either x_6 or x_7 produces a contradiction to connectivity or orthogonality, respectively. It is possible that x_3 and x_4 appear together again in some 4-circuit, D_2.

In that case, the only possibility is that $D_2 = \{x_3, x_4, x_6, x_7\}$. This collection of circuits satisfies all our assumptions, and therefore gives a matroid, which we label $M_{8,11}$. However, this structure also admits additional 4-circuits. A further 4-circuit, say D_3, must contain x_2 and one of x_3 or x_4. Within $M_{8,10}$, these last two elements are symmetric under the automorphism given by the permutation $(x_3, x_4)(x_5, x_8)(x_6, x_7)$, so we may assume $x_3 \in D_3$. Then $D_3 = \{x_2, x_3, x_6, x_8\}$. The inclusion of this circuit produces a second matroid in this case, which we label $M_{8,10^+}$. Further, this structure admits yet another 4-circuit, which must be $D_4 = \{x_2, x_4, x_5, x_7\}$. This third example we label $M_{8,10^{++}}$.

We may assume, then, that x_3 and x_4 do not appear in another 4-circuit outside of C_* and D_1. There must still be 4-circuits containing $\{x_3, x_6\}$ and $\{x_4, x_7\}$. Let these circuits be D_2 and D_3, respectively. In this case, x_2 is in each of these 4-circuits. In order to avoid an orthogonality contradiction with $E(M) - C_2$, one of x_5 and x_6 must be in D_2; and, in order to avoid an orthogonality contradiction with $E(M) - C_3$, one of x_6 and x_8 must be in D_2. Therefore, $D_2 = \{x_2, x_3, x_6, x_7\}$. Similar reasoning indicates $D_3 = \{x_2, x_4, x_5, x_7\}$. This collection of 4-circuits yields a matroid which permits no additional 4-circuits. We label this $M_{8,11}$.

Case 4.25.4. *Suppose* $n \leq 3$.

With this, we may assume that each element of C_* appears in at most three other 4-circuits, and that x_1 attains that maximum. This is the last major case, as in order for all the two-element subsets of $\{x_5, x_6, x_7, x_8\}$ to appear in a 4-circuit of M, at least six 4-circuits are required. We approach this last case in three phases of restricting the structure of the 4-circuits. First, we rule out the case when x_1 appears twice with the same element from C_* in two other 4-circuits. Next, we consider when x_1 appears with a certain element of $E(M) - C_*$ in three distinct 4-circuits, a case which produces one example. The final case yields no additional matroids, and concludes the search for eight-element matroids.

Subcase 4.25.4.1. x_1 *appears twice with the same element of* C_* *in two other 4-circuits.*

Suppose, without loss of generality, that x_1 and x_2 appear together in two 4-circuits in addition to C_*. It suffices to assume these are $C_1 = \{x_1, x_2, x_5, x_6\}$ and $C_2 = \{x_1, x_2, x_7, x_8\}$. As x_1 is in one more 4-circuit, say C_3, we may assume $\{x_1, x_3, x_5\} \subseteq C_3$, as x_3 and x_4

are symmetric, as are the elements of $E(M) - C_*$. Then $C_3 = \{x_1, x_3, x_5, x_7\}$ without loss of generality. Now, there is a 4-circuit, C_4, containing $\{x_6, x_8\}$. The remaining elements of C_4 come from $\{x_2, x_3, x_4\}$, and cannot be $\{x_2, x_4\}$, as $\{x_2, x_4, x_6, x_8\} = E(M) - C_3$. If $C_4 = \{x_2, x_3, x_6, x_8\}$, then all additional 4-circuits must contain x_3 and x_4, and x_3 may appear only once more. But the pairs $\{x_5, x_8\}$ and $\{x_6, x_7\}$ have yet to appear in a 4-circuit, a contradiction. Therefore $C_4 = \{x_3, x_4, x_6, x_8\}$. Further 4-circuits must contain either $\{x_2, x_4\}$ or $\{x_3, x_4\}$, and each of these may be used once. Therefore, the remaining circuits are either $\{x_2, x_4, x_5, x_8\}$ and $\{x_3, x_4, x_6, x_7\}$, or $\{x_2, x_4, x_6, x_7\}$ and $\{x_3, x_4, x_5, x_8\}$; each of these possibilities gives an orthogonality contradiction with $E(M) - C_4$. Thus x_1 cannot appear in two 4-circuits outside of C_* with the same element from C_*.

Subcase 4.25.4.2. *x_1 appears with a fixed element of $E(M) - C_*$ in three distinct 4-circuits.*

Without loss of generality, we may assume C_1, C_2, and C_3 each contain $\{x_1, x_5\}$. This implies $C_1 = \{x_1, x_2, x_5, x_6\}$, $C_2 = \{x_1, x_3, x_5, x_7\}$, and $C_3 = \{x_1, x_4, x_5, x_8\}$. There must be a 4-circuit, C_4, with $\{x_7, x_8\}$. This cannot be $\{x_3, x_4, x_7, x_8\} = E(M) - C_1$. Further, since the permutation $(x_3, x_4)(x_7, x_8)$ is an automorphism of M, we see that x_3 and x_4 are symmetric. Hence, we may assume $C_4 = \{x_2, x_3, x_7, x_8\}$ without loss of generality. There must also be 4-circuits C_5 and C_6 containing $\{x_6, x_7\}$ and $\{x_6, x_8\}$, respectively. By the restriction on pairs of elements in this case, this forces $C_5 = \{x_2, x_4, x_6, x_7\}$ and $\{x_3, x_4, x_6, x_8\}$. This collection of circuits satisfies all conditions on M. We label this matroid $M_{8,12}$.

Subcase 4.25.4.3. *Elements of C_* appear in a 4-circuit with an element $E(M) - C_*$ at most twice.*

Without loss of generality, we may assume the 4-circuits containing x_1 in this case are $C_1 = \{x_1, x_2, x_5, x_6\}$, $C_2 = \{x_1, x_3, x_5, x_7\}$, and $C_3 = \{x_1, x_4, x_6, x_8\}$. The pairs $\{x_5, x_8\}$, $\{x_6, x_7\}$, and $\{x_7, x_8\}$ must all appear in 4-circuits. The remaining elements of those 4-circuits come from $\{x_2, x_3, x_4\}$, with each pair of these occuring exactly once. Of these, if we consider the 4-circuit C_4 containing $\{x_3, x_4\}$, then we find that $C_4 = \{x_3, x_4, x_5, x_8\}$, as $\{x_3, x_4, x_7, x_8\} = E(M) - C_1$, and the pairs $\{x_5, x_8\}$ and $\{x_6, x_7\}$ are symmetric under the automorphism given by $(x_3, x_4)(x_5, x_6)(x_7, x_8)$. Now, x_3 must appear in a 4-circuit with x_6, so $C_5 = \{x_2, x_3, x_6, x_7\}$ must be a circuit. This implies $\{x_2, x_4, x_7, x_8\}$ is a circuit, but this is a contradiction, as then x_4 appears in three 4-circuits with x_8. Thus there are no matroids in this case, and our analysis of the eight-element matroids is complete.

To see that these matroids are all unique, we perform the following analysis, as in Proposition 4.24. Note that, by the structure of the cases in this argument, we need only be concerned with the matroids coming from Case 4.25.3. First, we distinguish the matroids using a count of their 4-circuits. Next, we assign an 8-tuple $(w_1, w_2, \ldots, w_8)$ to each matroid, where w_i is the number of distinct 4-circuits of M containing x_i. This is sufficient to determine the uniqueness of each matroid, as summarized in Figure 4.18. □

Matroid	4-Circuit #	Element Weights	Comment
$M_{8,10a}$	7	$(5,5,3,3,3,3,2,2)$	There are no elements with weight 4.
$M_{8,10b}$	7	$(5,4,4,3,3,3,2,2)$	There are elements of weight 4 and 2.
$M_{8,10b^+}$	8	$(5,5,4,4,4,3,3,2)$	There is an element of weight 2.
$M_{8,11}$	7	$(5,3,4,4,3,3,3,3)$	There are elements of weight 4, but none of weight 2.
$M_{8,11^+}$	8	$(5,4,5,4,3,4,3,4)$	The elements of weight 5 appear together only twice.
$M_{8,11^{++}}$	9	$(5,5,5,5,4,4,4,4)$	This matroid has nine 4-circuits.
$M_{8,12}$	8	$(5,5,4,4,4,3,4,3)$	The elements of weight 5 appear together three times.

Figure 4.18: Evidence for the uniqueness of each matroid determined in Proposition 4.25.

4.4 When M Has Exactly Nine Elements

The second major small-element case is when $|E(M)| = 9$. We first show that such a matroid must have rank 4, and then that it cannot have two disjoint 4-cocircuits. Finally, we determine all such matroids explicitly. In both the proof of Lemma 4.26 and Proposition 4.28, we will abuse the structure of the complements of 4-cocircuits of M. Specifically, we demonstrate that the corank of M restricted to the complement of a 4-cocircuit must be small, and thus narrow our search considerably. This technique will be utilized again in the ten-element case.

Lemma 4.26. *If $|E(M)| = 9$, then $r(M) = 4$.*

Proof. Suppose not. If $r(M) \leq 3$, then $r(M) = 3$ since M has 4-circuits. Further, each 4-element set must be a circuit, so $M \cong U_{3,9}$, which has no 4-cocircuit. Therefore $r(M) \geq 5$. Moreover, $r(M) = 5$, since M has 4-cocircuits and $r(M) \leq 6$ by dual reasoning to the above argument.

Let $E(M) = \{x_1, x_2, \ldots, x_9\}$, and consider a 4-cocircuit $D_1 = \{x_6, x_7, x_8, x_9\}$. If $X = E(M) - D_1$, then $r^*(M|X) = 1$. Further, the smallest cocircuits of $(M|X)^*$ have four elements. The only possbilities for $(M|X)^*$ are $U_{1,5}$ and $U_{1,4} \bigoplus U_{0,1}$, so we proceed in two cases.

First, assume that $(M|X)^* \cong U_{1,5}$. Then X contains no 4-circuits of M. Therefore, every circuit of M meets D_1, and must do so in at least two elements. There are $\binom{6}{2}$ distinct pairs of elements, each of which must be in some 4-circuit. These each meet one of the six distinct pairs of elements from D_1, and therefore some pair of elements from D_1 is used in at least two 4-circuits. Let C_1 and C_2 be those 4-circuits, and suppose, without loss of generality, that $C_1 \cap C_2 = \{x_6, x_7\}$. Then there is a circuit contained in $C_1 \cup C_2 - x_6$. This circuit cannot contain x_7, otherwise it violates orthogonality, and thus it is a 4-circuit contained in X, a contradiction. This implies $(M|X)^* \ncong U_{1,5}$.

Now suppose $(M|X)^* \cong U_{1,4} \bigoplus U_{0,1}$. In this case, $M|X$ contains exactly one 4-circuit which we label $C_1 = \{x_1, x_2, x_3, x_4\}$, without loss of generality. There are 4-circuits containing $\{x_1, x_5\}$, $\{x_2, x_5\}$, $\{x_3, x_5\}$, and $\{x_4, x_5\}$, and each such 4-circuit meets D_1 in two elements. Therefore, two of these 4-circuits must share one element from D_1. Without loss of generality, say $C_2 = \{x_1, x_5, x_6, x_7\}$ and $C_3 = \{x_2, x_5, x_6, x_8\}$. Let $S = C_2 \cup C_3$. Then $\lambda(S) = r(S) + r^*(S) - |S| \leq 4+4-6 = 2$, a contradiction. Thus $r(M) \neq 5$, and so the lemma is proved. $\square$

Lemma 4.27. *If $|E(M)| = 9$, then M has no two disjoint 4-cocircuits.*

Proof. Suppose not, letting D_1 and D_2 be a pair of disjoint 4-cocircuits. Let $\{e\} = E(M) - (D_1 \cup D_2)$. If $x \in D_1$, then there is a 4-circuit, C_1, containing $\{e, x\}$. This circuit must have at least two elements from D_1, and so must be disjoint from D_2. Let $\{y\} = D_1 - C_1$. Then there is a 4-circuit, C_2, containing $\{e, y\}$. As with C_1, we must have $|D_1 \cap C_2| = 3$. Then, there exists a circuit $C_3 \subseteq (C_1 \cup C_2) - e = D_1$, a contradiction. $\square$

The matroids in the statement of the following proposition are defined throughout the proof. As in the previous section, we can be assured that these are indeed matroids because the 4-circuits together with every 3-set that is in no 4-circuit form a 3-partition of $E(M)$, since no two such sets meet in more than two elements. This means that these sets are the hyperplanes of a paving matroid on $E(M)$, by Proposition 4.21.

Proposition 4.28. *Suppose M is a 4-connected matroid. If M has every element in a 4-cocircuit and every pair of elements in a 4-circuit, and $|E(M)| = 9$, then M is one of the following matroids: $M_{9,1}$, $M_{9,1a}$, $M_{9,1b}$, $M_{9,2}$, $M_{9,3}$, $M_{9,3^+}$, $M_{9,4}$, $M_{9,4^+}$, $M_{9,5}$, $M_{9,6}$.*

Proof. Let $E(M) = \{x_1, x_2, \ldots, x_9\}$, and consider a 4-cocircuit $D_1 = \{x_6, x_7, x_8, x_9\}$. If $X_1 = E(M) - D_1$, then $r^*(M|X_1) = 2$. Further, the smallest cocircuits of $(M|X_1)^*$ have four elements. This means that the only possibility for $(M|X_1)^*$ is $U_{2,5}$. Therefore, every subset of X_1 with four elements is a circuit of M, and so $M|X_1 \cong U_{3,5}$. This is true for every five-element hyperplane of M. With that in mind, consider a 4-cocircuit, D_2, containing x_1. This cocircuit must contain two additional elements from X_1 in order to avoid an orthogonality contradiction, and must also contain at least one element from D_1 by Lemma 4.27. Therefore, $D_2 = \{x_1, x_2, x_3, x_6\}$, without loss of generality. If we let $X_2 = E(M) - D_2$, then $M|X_2 \cong U_{3,5}$. There must also be a 4-cocircuit, D_3, containing x_4. As before, this cocircuit must contain three elements from X_1, and also three elements from X_2. Further, it must have at least one element from each D_1 and D_2. Therefore, without loss of generality, $D_3 = \{x_1, x_4, x_5, x_7\}$. Again, if we let $X_3 = E(M) - D_3$, then $X_3 \cong U_{3,5}$. Observe the symmetry between x_1, x_6, and x_7.

It should be noted that there may be no further 4-cocircuits of M, as there is no 4-element set that meets each of X_1, X_2, and X_3 in three elements. Therefore, all further hyperplanes of M have either three or four elements, which implies that all additional 4-circuits are hyperplanes and their complements cocircuits.

Case 4.28.1. *Suppose there is a 4-circuit, C_1, containing $\{x_1, x_6, x_7\}$.*

Without loss of generality, $C_1 = \{x_1, x_2, x_6, x_7\}$. Note that no other 4-circuit may contain $\{x_1, x_6, x_7\}$, or otherwise we get another local $U_{3,5}$, and, in turn, another 4-cocircuit. However, every additional 4-circuit must contain two of x_1, x_6, and x_7, in order to avoid an orthogonality contradiction with one of D_1, D_2, or D_3. In this case, consider a 4-circuit, C_2, containing x_3 and x_7. In order to avoid an orthogonality contradiction with one of D_1, D_2, or D_3, this circuit must contain either x_1 or x_6. These elements are symmetric under the automorphism given by the permutation $(x_1, x_6)(x_4, x_8)(x_5, x_9)$, so we may assume $x_1 \in C_2$. As C_2 must contain another element of D_1, we may assume $C_2 = \{x_1, x_3, x_7, x_8\}$, as x_8 and x_9 are symmetric.

Next, consider a 4-circuit, C_3, containing x_1 and x_9. Suppose $x_7 \in C_3$. In order to avoid an orthogonality contradiction with D_2, one of x_2 and x_3 must be in C_3, but these lead to an orthogonality contradiction with $E(M) - C_1$ or $E(M) - C_2$, respectively. Therefore $x_7 \notin C_3$, and one of x_4 and x_5 must be in C_3 in order to avoid an orthogonality contradiction with

D_3. These elements are symmetric, so we may assume $x_4 \in C_3$. Further, as the only element shared by D_1 and D_2 is x_6, it must be that $C_3 = \{x_1, x_4, x_6, x_9\}$ by orthogonality.

Now there is only one pair of elements not yet in a 4-circuit, and that is $\{x_5, x_6\}$. Let C_4 be 4-circuit containing $\{x_5, x_6\}$. As with the previous 4-circuits, C_4 must contain one of x_1 and x_7.

Subcase 4.28.1.1. *Suppose* $x_1 \in C_4$.

In order to avoid a contradiction, C_4 must contain another element from each of D_1, $E(M) - C_1$, and $E(M) - C_3$. Therefore $C_4 = \{x_1, x_5, x_6, x_8\}$. A matroid with this collection of circuits satisfies all our assumptions, and thus we get our first example, which we label $M_{9,1}$. It is possible, though, that additional 4-circuits exist as well as those noted above. Any such additional 4-circuit cannot contain x_1, as every four-element set with x_1 and one of x_6 and x_7 will meet some cocircuit in a single element. Now, if C_5 is another 4-circuit, it must be that $\{x_6, x_7\} \subseteq C_5$. Now, $x_2 \notin C_5$ because of orthogonality with $E(M) - C_1$, and so $x_3 \in C_5$, otherwise C_5 violates orthogonality with $E(M) - D_2$. The final element of C_5 must be either x_4 or x_5. Each of these gives rise to a distinct matroid. Let $M_{9,1a}$ be the matroid in which $C_5 = \{x_3, x_4 x_6, x_7\}$, and let $M_{9,1b}$ be the matroid in which $C_5 = \{x_3, x_5 x_6, x_7\}$. Neither of these matroids permits any additional 4-circuits.

We may now assume $x_1 \notin C_4$.

Subcase 4.28.1.2. $x_1 \notin C_4$.

In this case, C_4 must have another element from each of D_2 and $E(M) - C_1$. The only element they share is x_3, so $C_4 = \{x_3, x_5, x_6, x_7\}$. The inclusion of this circuit produces a matroid that satisfies our assumptions, and we label it $M_{9,2}$. Unlike the previous case, this structure admits no further 4-circuits. We know this matroid is distinct from $M_{9,1}$; because, x_1 is in four 4-circuits of $M_{9,1}$ not contained in X_1, X_2, or X_3, but each of x_1, x_6, or x_7 only appears three times in such 4-circuits of $M_{9,2}$. This completes the analysis of Case 4.28.1.

Case 4.28.2. *x_1, x_6, and x_7 do not appear together in a 4-circuit.*

Let C_1 be a 4-circuit containing x_1 and x_6. We may assume $C_1 = \{x_1, x_4, x_6, x_8\}$, without loss of generality.

Subcase 4.28.2.1. *x_1 and x_8 appear together in another 4-circuit.*

Let C_2 be an additional 4-circuit that contains $\{x_1, x_8\}$. Then $x_6 \notin C_2$, which means C_2 contains x_7. One of x_2 and x_3 must be in C_2, and so, as these elements are symmetric, we may assume $C_2 = \{x_1, x_2, x_7, x_8\}$. Next, consider a 4-circuit, C_3, containing x_1 and x_9. As x_6 and x_7 are symmetric under the automorphism given by the permutation $(x_2, x_4)(x_3, x_5)(x_6, x_7)$, we may assume $x_6 \in C_3$. In order to avoid an orthogonality contradiction with D_3 or $E(M) - C_1$, the final element of C_3 must be x_5, so $C_3 = \{x_1, x_5, x_6, x_9\}$. There must also be a 4-circuit, C_4, containing $\{x_3, x_7\}$. Either x_1 or x_6 is also on C_4, and this produces two cases.

Subcase 4.28.2.1.1. *Suppose $x_1 \in C_4$.*

Then $C_4 = \{x_1, x_3, x_7, x_9\}$, by orthogonality with one of D_1 or $E(M) - C_2$. In this case, the pair $\{x_6, x_7\}$ has yet to appear in a 4-circuit. Let C_5 be that circuit. To avoid an orthogonality contradiction, C_5 must have one element from each $\{x_2, x_3\}$ and $\{x_4, x_5\}$. Note that x_2 and x_3 are symmetric under an automorphism given by the permutation $(x_2, x_3)(x_4, x_5)(x_8, x_9)$. This automorphism also swaps x_4 and x_5, giving two distinct cases: either $C_5 = \{x_2, x_4, x_6, x_7\}$ or $C_5 = \{x_2, x_5, x_6, x_7\}$. Both cases satisfy our assuptions for M, and thus yield matroids which we label $M_{9,3}$ and $M_{9,4}$, respectively. Further, these sets of circuits permit one additional 4-circuit in each case. If we let $M_{9,3^+}$ be the matroid with all the 4-circuits of $M_{9,3}$ and also $\{x_3, x_5, x_6, x_7\}$, we find another example. Similarly, we get a fourth example from a matroid with all the 4-circuits $M_{9,4}$ together with $\{x_3, x_4, x_6, x_7\}$, which we label $M_{9,4^+}$. Evidently, $M_{9,4^+}$ is distinct from $M_{9,3^+}$, as they have all 4-circuits in common except one. No further 4-circuits may be added to these latter examples without contradicting either connectivity or orthogonality.

Subcase 4.28.2.1.2. *Suppose $x_1 \notin C_4$.*

This implies that C_4 contains x_6. Now, C_4 must also contain one of x_4 and x_5 in order to avoid an orthogonality contradiction with D_3. These elements are not symmetric; however, the two choices of C_4 produce sets of 4-circuits that are symmetric under the automorphism given by the permutation $(x_1, x_6)(x_2, x_3)(x_4, x_8)(x_5, x_9)$. Therefore, we may assume $C_4 = \{x_3, x_4, x_6, x_7\}$. With this, we have a matroid satisfying all assumptions, which we label $M_{9,5}$. This structure admits one further possible 4-circuit. Note that such a 4-circuit, say C_5, must not contain $\{x_1, x_6\}$, as, in order to avoid an orthogonality contradiction with D_1, such a circuit must also contain one of x_8 and x_9, but then would meet either C_1 or C_3 in three elements. Also, C_5 cannot contain $\{x_1, x_7\}$, as then $C_5 = \{x_1, x_3, x_7, x_9\}$ is forced, and we addressed this circuit in the previous case. Therefore, $x_1 \notin C_5$, and C_5 must contain both x_6 and x_7. As with C_4, this implies that one element from each $\{x_2, x_3\}$ and $\{x_4, x_5\}$ is in C_5. Therefore, since C_5 does not have three common elements with C_4, we get $C_5 = \{x_2, x_5, x_6, x_7\}$. The addition of this 4-circuit to the set of circuits produces a second matroid satisfying our hypotheses; however, the resulting matroid is isomorphic to $M_{9,3}$ under the automorphism given by the permutation $(1, 6)(4, 9)(5, 8)$. Therefore, we close this case having found one additional matroid.

Subcase 4.28.2.2. *x_1 and x_8 do not appear together again in any further 4-circuits of M.*

We may generalize this assumption and say that each pair of elements, one from $\{x_1, x_6, x_7\}$ and the other from $E(M) - \{x_1, x_6, x_7\}$, appears at most once in a 4-circuit outside of the local $U_{3,5}$ structures of M. As a reminder to the reader, we still have $C_1 = \{x_1, x_4, x_6, x_8\}$ in this case. Let C_2 be a 4-circuit containing $\{x_2, x_7\}$. It must be that one of x_1 and x_6 is in C_2, and these elements are symmetric. Therefore, we may assume $x_1 \in C_2$. In order to not contradict orthogonality, one of x_8 and x_9 must be in C_2. By our case assumption, C_2 must contain x_9. Therefore, $C_2 = \{x_1, x_2, x_7, x_9\}$. There must also be a 4-circuit, C_3, containing

$\{x_5, x_6\}$. Again, in order to avoid an orthogonality contradiction with D_1, one element from $\{x_7, x_8, x_9\}$ must be in C_3. Therefore it must be that $x_7 \in C_3$, since, if not, then $x_1 \in C_3$, in which case C_3 may contain neither x_8 nor x_9 by the case assumption. Further, in order to avoid contradicting orthogonality with D_2, one of x_2 and x_3 must be in C_3. Again, by our case assumption, this implies $C_3 = \{x_3, x_5, x_6, x_7\}$, since x_2 and x_7 appear together in C_2. This gives a set of 4-circuits that satisfy all our assumptions. Label the matroid with these 4-circuits $M_{9,6}$. It is easy to see that our condition on this case prohibits the addition of any further 4-circuits to this list. Thus our analysis of 9-element matroids is complete.

□

4.5 When M Has Exactly Ten Elements

This section closely resembles the nine-element case in the organization of its arguments. We begin by determining the rank of a ten-element matroid with property $(P2)$, and proceed to show that it cannot have two disjoint 4-cocircuits. We then restrict the structure of the complements of 4-cocircuits, and finally prove that the only matroid with property $(P2)$ on ten elements is the well-known R_{10}.

Lemma 4.29. *If $|E(M)| = 10$, then $r(M) = 5$.*

Proof. Clearly $4 \leq r(M) \leq 6$, as M is 4-connected.

If $r(M) = 4$, then the complement of any 4-cocircuit is a 6-point plane. There must be a 4-cocircuit using an element of that plane. Such a cocircuit must be contained in that plane in order to avoid an orthogonality contradiction. But then that 4-cocircuit is also a 4-circuit, a contradiction. Thus $r(M) \neq 4$.

The case in which $r(M) = 6$ leads to contradiction by a similar dual argument. □

Lemma 4.30. *If $|E(M)| = 10$, then M has no two disjoint 4-cocircuits.*

Proof. Let $E(M) = \{x_1, x_2, \ldots, x_{10}\}$. Suppose the lemma fails, and let D_1 and D_2 be disjoint 4-cocircuits of M. We may assume $D_1 = \{x_1, x_2, x_3, x_4\}$ and $D_2 = \{x_5, x_6, x_7, x_8\}$. By Proposition 4.6, we know $M|(D_1 \cup D_2) \cong M(K_{2,4})$. Without loss of generality, let $\{x_1, x_5\}$, $\{x_2, x_6\}$, $\{x_3, x_7\}$, and $\{x_4, x_8\}$ be the pairs that appear together in the 4-circuits of $M|(D_1 \cup D_2)$. The rank of each 6-element set comprised of three of the aforementioned pairs is 4, therefore making it a hyperplane. This gives us cocircuits $D_3 = \{x_1, x_5, x_9, x_{10}\}$, $D_4 = \{x_2, x_6, x_9, x_{10}\}$, $D_5 = \{x_3, x_7, x_9, x_{10}\}$, and $D_6 = \{x_4, x_8, x_9, x_{10}\}$. Consider a 4-circuit, C, containing x_1 and x_9. In order to avoid an orthogonality contradiction with one of these four 4-cocircuits, it must be that $x_{10} \in C$. In order to avoid a similar contradiction with D_1, we may assume, without loss of generality, that $x_2 \in C$. Circuit elimination on D_5 and D_6 indicates that there is a cocircuit contained in $\{x_3, x_4, x_5, x_7, x_8\} = (D_5 \cup D_6) - x_6$. This cannot contain x_5, otherwise we get an orthogonality contradiction with C. But then $\{x_3, x_4, x_7, x_8\}$ is both a circuit and a cocircuit, contradicting the 4-connectivity of M. Thus M has no two disjoint 4-cocircuits. □

In a simple matroid, we say that a point is *doubled* if that element is replaced by two elements in parallel.

Lemma 4.31. *Suppose* $|E(M)| = 10$. *If* X *is the complement of a 4-cocircuit of* M*, then* $(M|X)^* \cong T^2$*, where* T^2 *is the matroid* $U_{2,3}$ *with every point doubled.*

Proof. Let $E(M) = \{x_1, x_2, \ldots, x_{10}\}$, and consider a 4-cocircuit $D_1 = \{x_7, x_8, x_9, x_{10}\}$. If $X_1 = E(M) - D_1$, then by Lemma 4.29 we have $r^*(M|X_1) = 2$. Further, the smallest cocircuits of $(M|X_1)^*$ have four elements. As possibilities for $(M|X_1)^*$, there are five rank-2 6-element matroids with cocircuits having at least 4 elements: $U_{2,6}$, $U_{2,5} \oplus U_{0,1}$, $U_{2,5}$ with one point doubled, $U_{2,4}$ with two points doubled, and T^2. We address these cases in order.

Case 4.31.1. *Suppose* $(M|X_1)^* \cong U_{2,6}$.

Now there are no 4-circuits contained in X_1. Every pair of elements of X_1 is in some 4-circuit. Each such 4-circuit must contain two elements of D_1 to avoid a contradiction to orthogonality. There are 15 distinct pairs in X_1 and only 6 distinct pairs in D_1. Therefore there are two 4-circuits, say C_1 and C_2 such that $C_1 \cap D_1 = C_2 \cap D_1$. Let e be one of the elements in $C_1 \cap D_1$. Then the circuit contained in $(C_1 \cup C_2) - e$ is fully contained in X, a contradiction. Therefore $(M|X_1)^* \ncong U_{2,6}$.

Case 4.31.2. *Suppose* $(M|X_1)^* \cong U_{2,5} \oplus U_{0,1}$.

Now $M|X_1 \cong U_{3,5} \oplus U_{1,1}$. Suppose x_1 is the element corresponding to the $U_{1,1}$-component. There must be a 4-cocircuit, D_2, containing x_1. This cocircuit must contain some element of D_1 by Lemma 4.30, but must also contain some element of $X_1 - x_1$, by Lemma 4.2. But every 4-element subset of $X_1 - x_1$ is a circuit, so, in order to avoid an orthogonality contradiction, any cocircuit meeting $X_1 - x_1$ must do so in at least theree elements. This requires D_2 to have at least five elements, a contradiction. Therefore $(M|X_1)^* \ncong U_{2,5} \oplus U_{0,1}$.

Case 4.31.3. *Suppose* $(M|X_1)^*$ *is isomorphic to* $U_{2,5}$ *with one point doubled.*

In this case, $(M|X_1) \cong U_{3,5} \oplus_2 U_{1,3}$. Suppose $\{x_1, x_2, x_3, x_4\}$ corresponds to the $U_{3,4}$-component of $M|X_1$; then $\{x_5, x_6\}$ corresponds to the $U_{1,2}$-component. There is a 4-circuit containing $\{x_i, x_j\}$ for every pair with $i \in \{1, 2, 3, 4\}$, and $j \in \{5, 6\}$. These 4-circuits cannot be contained in X_1, and so must have two elements from D_1. There are eight such pairs from X_1, and only six distinct pairs of elements from D_1, so, again, some pair from D_1 must occur twice in these 4-circuits. This leads to a contradiction as in the first case.

Case 4.31.4. *Suppose* $(M|X_1)^*$ *be isomorphic to* $U_{2,4}$ *with two points doubled.*

This is the most lengthy case, and will require several subcases of analysis. We know $M|X_1$ is isomorphic to $U_{2,4}$ with two copies of $U_{2,3}$ 2-summed at different points. Thus, there are exactly two 4-circuits contained in X_1, and these share two elements. Without loss of generality, we may assume those circuits are $C_1 = \{x_1, x_2, x_5, x_6\}$ and $C_2 = \{x_3, x_4, x_5, x_6\}$. Consider a 4-cocircuit, D_2, containing x_5.

Subcase 4.31.4.1. *Assume x_5 and x_6 do not appear together in a 4-cocircuit.*

If $x_6 \notin D_2$, then D_2 must have one element from each of $\{x_1, x_2\}$ and $\{x_3, x_4\}$. Without loss of generality, say $\{x_1, x_3\} \subseteq D_2$. By Lemma 4.30, D_1 and D_2 must share an element; therefore, we may assume $D_2 = \{x_1, x_3, x_5, x_7\}$. Consider, then, the 4-circuits C_3 and C_4 containing $\{x_1, x_4\}$ and $\{x_2, x_3\}$, respectively. These circuits are not contained in X_1, and therefore must have two elements from D_1. In order to avoid an orthogonality contradiction with D_2, both C_3 and C_4 must contain x_7. Therefore, we may assume $C_3 = \{x_1, x_4, x_7, x_8\}$ and $C_4 = \{x_2, x_3, x_7, x_9\}$. Then, consider a 4-circuit, C_5, containing x_2 and x_4. Again, C_5 must have two elements from D_2, and, evidently, $x_7 \notin C_5$. If $C_5 = \{x_2, x_4, x_8, x_9\}$, then $r(\{x_1, x_2, x_3, x_4, x_7, x_8, x_9\}) = 4$, and M has a 3-cocircuit, a contradiction. Therefore, without loss of generality, $C_5 = \{x_2, x_4, x_8, x_0\}$. In this case, $r(C_3 \cup C_5) = 4$, and $\{x_3, x_5, x_6, x_9\} = E(M) - (C_3 \cup C_5)$ is a cocircuit containing both x_5 and x_6, a contradiction. We now know that we may assume that

Subcase 4.31.4.2. *D_2 contains $\{x_5, x_6\}$.*

From here we are able to systematically determine all circuits and cocircuits of M until we arrive at a contradiction. Circuit elimination on C_1 and C_2 indicates that both $C' = \{x_1, x_2, x_3, x_4, x_5\}$ and $C'' = \{x_1, x_2, x_3, x_4, x_6\}$ are circuits, so D_2 must contain an element from $\{x_1, x_2, x_3, x_4\}$. Without loss of generality, we may assume $D_2 = \{x_1, x_5, x_6, x_7\}$. We now suppose C_3 and C_4 are 4-circuits containing $\{x_1, x_3\}$ and $\{x_1, x_4\}$, respectively. Each of these must contain x_7, otherwise we get an orthogonality contradiction. Therefore, it suffices to let $C_3 = \{x_1, x_3, x_7, x_8\}$ and $C_4 = \{x_1, x_4, x_7, x_9\}$. Then $r(C_3 \cup C_4) = 4$, and $D_3 = \{x_2, x_5, x_6, x_{10}\} = E(M) - (C_3 \cup C_4)$ is a cocircuit. Circuit elimination on D_2 and D_3 produces an additional 4-cocircuit, $D_4 = \{x_1, x_2, x_7, x_{10}\}$. We also get a 4-circuit $C_5 \subseteq$

$(C_3 \cup C_4) - x_1$. This circuit cannot contain x_7, otherwise it violates orthogonality with D_2, so $C_5 = \{x_3, x_4, x_8, x_9\}$.

Now we may determine 4-circuits C_6 and C_7 containing $\{x_2, x_3\}$ and $\{x_2, x_4\}$, respectively. Neither of these may contain x_7, otherwise a pair of elements from D_1 is shared by at least two 4-circuits and we may find an extra 4-circuit in X_1, a contradiction as in the previous cases. Therefore, x_{10} is in both of these circuits. This presents two possibilities: either $x_8 \in C_6$ and $x_9 \in C_7$, or vice versa. In the former case, $r(C_3 \cup C_6) = 4 = r(C_4 \cup C_7)$, so $\{x_4, x_5, x_6, x_9\} = E(M) - (C_3 \cup C_6)$ and $\{x_3, x_5, x_6, x_8\} = E(M) - (C_4 \cup C_7)$ are cocircuits. This is a contradiction to Lemma 4.10, as $\{x_5, x_6\} = D_2 \cap D_3$. Therefore, we get $C_6 = \{x_2, x_3, x_9, x_{10}\}$ and $C_7 = \{x_2, x_4, x_8, x_{10}\}$.

Consider, now, a 4-cocircuit D_5 containing x_3. If $x_1 \notin D_5$, we may assume $x_2 \notin D_5$, by the symmetry of these elements under the automorphism given by the permutation $(x_1, x_2)(x_7, x_{10})(x_8, x_9)$. In this case, D_5 must contain an element from each of $\{x_4, x_5\}$, $\{x_4, x_6\}$, $\{x_7, x_8\}$, and $\{x_9, x_{10}\}$ in order to avoid an orthogonality contradiction with C', C'', C_3, and C_6, respectively. Therefore $x_4 \in D_5$. Also, as M has no two disjoint 4-cocircuits, D_5 must contain one of x_7 and x_{10} so as to meet D_4. This forces $D_5 = \{x_3, x_4, x_7, x_{10}\}$.

The last 4-circuit we will determine is C_8, containing $\{x_5, x_8\}$. If $x_{10} \in C_8$, then $C_8 = \{x_1, x_5, x_8, x_{10}\}$, to avoid orthogonality contradictions with D_2 and D_4. But then, circuit elimination with C_7 and C_8 forces $\{x_1, x_2, x_4, x_5\} \subseteq X_1$ to be a circuit, a contradiction. Therefore, it must be that $x_{10} \notin C_8$. Note, also, that $x_2 \notin C_8$, as then C_8 cannot avoid meeting one of D_1, D_2, or D_5 in a single element. Therefore, in order to avoid an orthogonality contradiction with D_3, we have $x_6 \in C_8$. This forces $x_9 \in C_8$, otherwise we get a similar contradiction with one of D_1 or D_5. Therefore $C_8 = \{x_5, x_6, x_8, x_9\}$. Then, $r(C_1 \cup C_8) = 4$, which implies $\{x_3, x_4, x_7, x_{10}\}$ is a cocircuit. This contradicts Lemma 4.10, as $\{x_7, x_{10}\} = D_1 \cap D_4$. Thus $(M|X_1)^*$ must not be isomorphic to $U_{2,4}$ with two points doubled. The only remaining possibility is that $(M|X_1)^* \cong T^2$, as desired. $\square$

We now have all the tools necessary to determine the lone matroid on ten elements.

Proposition 4.32. *Suppose M is a 4-connected matroid. If M has every element in a 4-cocircuit and every pair of elements in a 4-circuit, and $|E(M)| = 10$, then $M \cong R_{10}$.*

Proof. Let $E(M) = \{x_1, x_2, \ldots, x_{10}\}$, and suppose $D_1 = \{x_7, x_8, x_9, x_{10}\}$ is a cocircuit of M. Then, by Lemma 4.31, $(M\backslash D_1)^* \cong T^2$, and we get circuits $C_1 = \{x_1, x_2, x_3, x_4\}$, $C_2 = \{x_1, x_2, x_5, x_6\}$, and $C_3 = \{x_3, x_4, x_5, x_6\}$, without loss of generality. Further, a 4-cocircuit, D_2, containing x_1 may be assumed to be $D_2 = \{x_1, x_2, x_7, x_8\}$ by orthogonality. Then $(M\backslash D_2)^* \cong T^2$. The elements x_9 and x_{10} either appear together or not at all in all the 4-circuits contained in $M\backslash D_2$. One of these 4-circuits is C_3. If we let C_4 and C_5 be the other two 4-circuits, we get two possibilities: either $C_4 = \{x_3, x_4, x_9, x_{10}\}$ and $C_5 = \{x_5, x_6, x_9, x_{10}\}$, or, without loss of generality, $C_4 = \{x_3, x_5, x_9, x_{10}\}$ and $C_5 = \{x_4, x_6, x_9, x_{10}\}$. In the first case, $r(C_1 \cup C_4) = 4$, so $\{x_5, x_6, x_7, x_8\} = E(M) - (C_1 \cup C_4)$ is a cocircuit, a contradiction to Lemma 4.10 as $\{x_7, x_8\} = D_1 \cap D_2$.

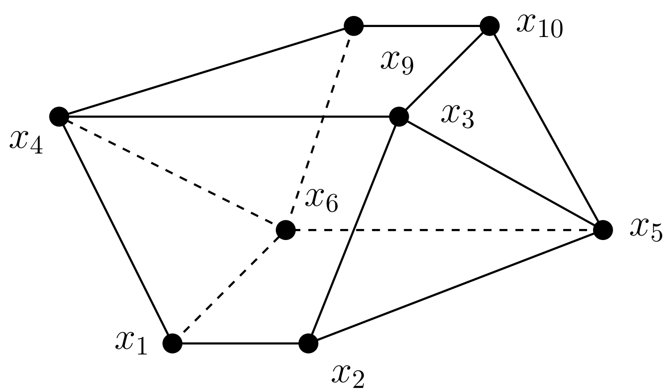

Figure 4.19: A forbidden configuration of 4-circuits when $|E(M) = 10|$.

Therefore, we get the circuits in the latter case, and rule out the previous configuration, depicted in Figure 4.19 in all further instances when two 4-cocircuits share two elements. That is, we need only consider matroids that do not have the as a restriction the matroid depicted in Figure 4.19. Consider a 4-cocircuit, D_3, containing x_3.

Claim 4.32.1. *M has a 4-cocircuit containing $\{x_3, x_4\}$*

Suppose not. The elements x_4 and x_6 are symmetric under the automorphism given by the permutation $(x_1, x_9)(x_2, x_{10})(x_4, x_6)$, so we assume further that there is no 4-cocircuit containing $\{x_3, x_6\}$. Then, in order to avoid an orthogonality contradiction with C_3, we get $x_5 \in D_3$. Additionally, D_3 must have one element from each $\{x_1, x_2\}$ and $\{x_9, x_0\}$ to avoid an orthogonality contradiction with C_1 and C_4, respectively. The elements within each of these pairs are symmetric, so we may assume $D_3 = \{x_1, x_3, x_5, x_9\}$. Then $(M \backslash D_3)^* \cong T^2$. In order to avoid orthogonality contradictions with D_1 and D_2, the 4-circuits contained in $M \backslash D_3$ must always contain two elements of $\{x_2, x_7, x_8\}$ and $\{x_7, x_8, x_{10}\}$. Therefore, without loss of generality, we get circuits $C_6 = \{x_2, x_4, x_7, x_{10}\}$, $C_7 = \{x_2, x_6, x_8, x_{10}\}$, and $C_8 = \{x_4, x_6, x_7, x_8\}$. But then, $r(C_2 \cup C_7) = 4$, and $\{x_3, x_4, x_7, x_9\} = E(M) - (C_2 \cup C_7)$ is a 4-cocircuit containing $\{x_3, x_4\}$, a contradiction. This proves our claim.

It must be, then, that there is a 4-cocircuit containing both x_3 and x_4. We may suppose D_3 is such a cocircuit. In order to avoid an orthogonality contradiction with C_4 or C_5, there must be a second element of each C_4 and C_5 in D_3. We argue that

Claim 4.32.2. *$x_9 \in D_3$.*

Suppose $x_5 \in D_3$. Then, as M has no disjoint 4-cocircuits, $x_7 \in D_3$. But then D_3 only meets C_5 in one element, a contradiction. A similar argument shows $x_6 \notin D_3$. Hence, we may assume $x_9 \in D_3$. The final element of D_3 must come from D_2, and cannot be either x_1 or x_2, otherwise D_3 violates orthogonality with C_2. Therefore, without loss of generality, $D_3 = \{x_3, x_4, x_7, x_9\}$.

This implies $(M \backslash D_3)^* \cong T^2$. Since D_1 and D_3 share two elements, we can argue as before on D_1 and D_2. Therefore, we may assume, without loss of generality, that the 4-circuits contained in $M \backslash D_3$ are $C_6 = \{x_1, x_5, x_8, x_{10}\}$, $C_7 = \{x_2, x_6, x_8, x_0\}$, and C_2. Now we know $r(C_5 \cup C_6) = 4$, so $D_4 = \{x_2, x_3, x_6, x_7\} = E(M) - (C_5 \cup C_6)$ is a 4-cocircuit. Similarly, $D_5 = \{x_1, x_4, x_5, x_7\} = E(M) - (C_4 \cup C_7)$ is a cocircuit. In turn, these force 4-circuits

$C_8 = \{x_1, x_4, x_8, x_9\}$ and $C_9 = \{x_2, x_3, x_8, x_9\}$ in the local T^2-structure of their complements. Then, $r(C_8 \cup C_9) = 4$, so $D_6 = \{x_5, x_6, x_7, x_{10}\}$.

Note that there is not yet a 4-circuit containing x_1 and x_7. Such a 4-circuit, say C_{10}, must contain a second element from each of D_1, D_3, D_4, and D_6, in order to avoid an orthogaonlity contradiction. Noting the automorphism given by the permutation $(x_3, x_6)(x_4, x_5)(x_9, x_{10})$, we may assume $C_{10} = \{x_1, x_3, x_7, x_{10}\}$. This last 4-circuit will allow us to determine a hyperplane, which determines a 4-cocircuit, which determines a local T^2, which, in turn, determines a further 4-circuit, which then allows this process to repeat until all 4-circuits and 4-cocircuits of M are determined. These are all determined explicitly, and no further assumptions are necessary. In the list that follows, we maintain the convention that those sets labeled D_i represent cocircuits, while those labeled C_i represent circuits. We list these in the sequence that they may be determined, without further comment: $D_7 = \{x_5, x_6, x_8, x_9\}$, $C_{11} = \{x_2, x_4, x_7, x_{10}\}$, $D_8 = \{x_2, x_4, x_6, x_9\}$, $C_{12} = \{x_1, x_6, x_7, x_9\}$, $D_9 = \{x_2, x_4, x_6, x_9\}$, $C_{13} = \{x_3, x_5, x_7, x_8\}$, $D_{10} = \{x_1, x_3, x_6, x_8\}$, $C_{14} = \{x_2, x_5, x_7, x_9\}$, $D_{11} = \{x_1, x_3, x_5, x_9\}$, $C_{15} = \{x_4, x_6, x_7, x_8\}$. We also get $D_{12} = \{x_2, x_3, x_5, x_{10}\}$, $D_{13} = \{x_1, x_2, x_9, x_{10}\}$, $D_{14} = \{x_1, x_4, x_6, x_{10}\}$, and $D_{15} = \{x_3, x_4, x_8, x_{10}\}$.

$$A = \begin{array}{c} \begin{matrix} y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 & y_9 & y_{10} \end{matrix} \\ \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix} \end{array}$$

Figure 4.20: The matrix A.

This provides a complete list of all 4-circuits and 4-cocircuits of M. Let $M' = M(A)$ for the matrix A in Figure 4.20. Evidently, $M' \cong R_{10}$. Let $\phi : E(M') \to E(M)$ be a map given by $\phi(y_i) = x_i$. Then ϕ is a weak map, and an application of Theorem 3.6 concludes our proof. $\square$

4.6 When M Has More Than 10 and Fewer Than 16 elements

In this section, we find only two examples: one when M has 12 elements and the other when M has 14 elements. We show, first, that M cannot have exactly 11, 13, or 15 elements.

Proposition 4.33. *If M has property $(P2)$, then $|E(M)| \neq 11$.*

Proof. Let $E(M) = \{x_1, x_2, \ldots, x_{11}\}$. By Proposition 4.16, we know M has two disjoint 4-cocircuits. We may assume $D_1 = \{x_1, x_2, x_3, x_4\}$ and $D_2 = \{x_5, x_6, x_7, x_8\}$ are those cocircuits. By Proposition 4.6, we have $M|(D_1 \cup D_2) \cong M(K_{2,4})$. We may assume that $\{x_1, x_5\}$, $\{x_2, x_6\}$, $\{x_3, x_7\}$, and $\{x_4, x_8\}$ are the series pairs in $M|(D_1 \cup D_2)$. By $(P2)$, there is a 4-cocircuit D_3 that contains x_9. By orthogonality, $|D_3 \cap (D_1 \cup D_2)| = 2$, and so we may assume $D_3 = \{x_1, x_5, x_9, x_{11}\}$. Similarly, there is a 4-cocircuit D_4 containing x_{10}. By Proposition 4.2, we know $\{x_1, x_5\} \not\subseteq D_4$; therefore $D_4 = \{x_2, x_6, x_{10}, x_{11}\}$. Now the basic structure of the 4-cocircuits has been determined, and is depicted in Figure 4.21. Next, we will show that

Claim 4.33.1. $r(M) = 5$.

Since $M|(D_1 \cup D_2) \cong M(K_{2,4})$ and $r(M(K_{2,4})) = 5$, we know that $r(M) \geq 5$. If $r(M) > 5$, then $D_1 \cup D_2$ is contained in a hyperplane H of M. But then M has a cocircuit of size $|E(M) - H| \leq 3$, a contradiction to 4-connectivity. Therefore $r(M) = 5$.

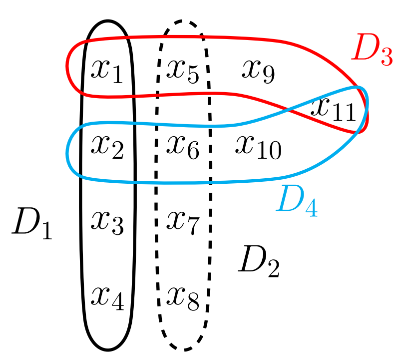

Figure 4.21: A set diagram of the 4-cocircuits in M.

We need two additional 4-circuits in order to produce a contradiction. By $(P2)$, we have a 4-circuit C_1 containing $\{x_9, x_11\}$. By orthogonality, $|C_1 \cap D_4| = 2$, so we may assume $x_6 \in C_1$. Similarly, $|C_1 \cap D_2| = 2$, so without loss of generality, either $x_5 \in C_1$ or $x_7 \in C_1$. We assert that, possibly with some relabeling,

Claim 4.33.2. $\{x_6, x_7, x_9, x_{11}\}$ *is a circuit.*

Note that, by symmetry, we are satisfied to find a 4-cocircuit containing x_{11} that meets both $\{x_9, x_{10}\}$ and $\{x_3, x_4, x_7, x_8\}$. Therefore, $C_1 = \{x_5, x_6, x_9, x_{11}\}$. By circuit elimination with $\{x_1, x_2, x_5, x_6\}$, we get that $C_2 = \{x_1, x_2, x_9, x_{11}\}$ is a circuit. Similarly, there is a 4-circuit C_3 containing $\{x_{10}, x_{11}\}$. Applying the same reasoning as before, we may assume that $C_3 = \{x_5, x_6, x_{10}, x_{11}\}$, without loss of generality. But now there must be a circuit $C_4 \subseteq (C_1 \cup C_3) - x_5 = \{x_6, x_9, x_{10}, x_{11}\}$. By orthogonality, $x_6 \notin C_4$, but then $|C_4| = 3$, a contradiction. Therefore the claim holds.

Now we may assume $C_1 = \{x_6, x_7, x_9, x_{11}\}$ is a circuit. Given that D_3 is a cocircuit, we have $r(M \backslash D_3) = 4$. Therefore, since $\{x_2, x_6, x_7, x_8\}$ is independent in M by orthogonality, it must be that $\{x_2, x_6, x_7, x_8\}$ spans $M \backslash D_3$. Hence there is a circuit C_2 contained in $\{x_2, x_6, x_7, x_8\} \cup \{x_{10}\}$ that must contain x_{10}. By orthogonality, $x_2 \notin C_2$, so $C_2 = \{x_6, x_7, x_8, x_{10}\}$.

By circuit elimination, there is a cocircuit $D' \subseteq (D_1 \cup D_4) - \{x_2\} = \{x_1, x_3, x_4, x_6, x_{10}, x_{11}\}$. We know $|D'| \geq 4$; therefore, D' meets $\{x_1, x_3, x_4, x_6\}$. Given the circuits in $M|(D_1 \cup D_2)$, it must be that $\{x_1, x_3, x_4, x_6\} \subseteq D'$, by othogonality. Further, since x_6 is in D', so too must x_{10} and x_{11} be, by orthogonality with C_2 and C_1, respectively. Hence $D' = \{x_1, x_3, x_4, x_6, x_{10}, x_{11}\}$, and so $E(M) - D' = \{x_2, x_5, x_7, x_8, x_9\}$ is a circuit hyperplane. This circuit-hyperplane violates orthogonality with D_1, and this contradiction proves the proposition.

□

Proposition 4.34. *If M has property $(P2)$, then $|E(M)| \neq 13$ and $|E(M)| \neq 15$.*

Proof. In both cases, we may assume M has three pairwise-disjoint 4-cocircuits, D_1, D_2, and D_3, forming a local $K_{3,4}$-structure, by Proposition 4.19 and Lemma 4.17. Let the elements these sets be $\{x_1, x_2, x_3, x_4\}$, $\{x_5, x_6, x_7, x_8\}$, and $\{x_9, x_{10}, x_{11}, x_{12}\}$, respectively. We may assume the circuits of $M|(D_1 \cup D_2 \cup D_3)$ are as they appear in Figure 4.22.

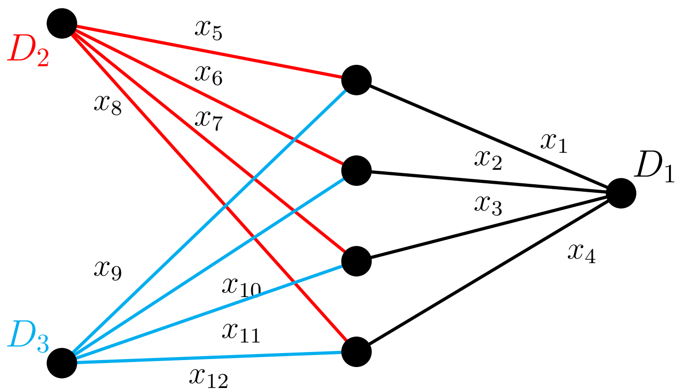

Figure 4.22: The graph $K_{3,4}$ provides structure to $M|(D_1 \cup D_2 \cup D_3)$.

Assume $|E(M)| = 13$, and let x_{13} be the element of M not in D_1, D_2, or D_3. Then x_{13} is in a 4-cocircuit, D_4, which must meet each of D_1, D_2, and D_3. Without loss of generality, $D_4 = \{x_1, x_5, x_9, x_{13}\}$. Consider a 4-circuit, C_1, containing $\{x_2, x_{13}\}$. This must contain a second element from each of D_1 and D_4, and is therefore disjoint from D_2 and D_3. Therefore, it suffices to assume $C_1 = \{x_1, x_2, x_3, x_{13}\}$. Similarly, a 4-circuit, C_2, containing $\{x_4, x_{13}\}$ must contain x_1 and one of $\{x_2, x_3\}$. But, then $D_1 \subseteq (C_1 \cup C_2) - \{x_{13}\}$ contains a circuit, a contradiction. Hence, $|E(M)| \neq 13$.

Assume, then, that $|E(M)| = 15$. Now, we have three elements not in D_1, D_2, or D_3, call them x_{13}, x_{14}, and x_{15}. Each of these is in a 4-cocircuit, which we may assume are $D_4 = \{x_1, x_5, x_9, x_{13}\}$, $D_5 = \{x_2, x_6, x_{10}, x_{14}\}$, and $D_4 = \{x_3, x_7, x_{11}, x_{15}\}$, respectively. Note that this implies $M|(D_4 \cup D_5 \cup D_6) \cong M(K_{3,4})$. Consider a 4-circuit, C_1, containing x_4 and x_{13}. In order to avoid an orthogonality contradiction, this circuit must contain a second element from each D_1 and D_4, and is therefore disjoint from all other 4-cocircuits. But every element of M, save x_1, is in some other 4-cocircuit, a contradiction. Thus $|E(M)| \neq 15$, proving the proposition. $\square$

Next, we introduce matroids on 12 and 14 elements, which we call $M_{12} \cong M(P)$ and $M_{14} \cong M(Q)$. The matrix entries are over $GF(4)$, where every element is its additive inverse and $\alpha^2 + \alpha + 1 = 0$. We proceed to prove that these are the unique matroids of their respective sizes with property $(P2)$. The proofs for each proposition are similar, although the 14-element case is more lengthy.

$$P = \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_9 & x_6 & x_7 & x_8 & x_{10} & x_{11} & x_{12} \end{matrix} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & \alpha & 1 & \alpha \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & \alpha^2 & 1 & \alpha^2 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

Figure 4.23: The matrix P.

$$Q = \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_9 & x_6 & x_7 & x_8 & x_{10} & x_{11} & x_{12} & x_{13} & x_{14} \end{matrix} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & \alpha & \alpha \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & \alpha^2 & \alpha^2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

Figure 4.24: The matrix Q.

Proposition 4.35. *Let $|E(M)| = 12$. Then M is a 4-connected matroid in which every element is in a 4-cocircuit and every pair of elements in a 4-circuit if and only if $M \cong M_{12}$.*

Proof. Clearly, $r(M(P)) = 5$. It is straightforward to verify that $M(P)$ is 4-connected and has property $(P2)$.

Now, suppose that M satisfies the given conditions. By Corollary **??**, M has two pairs of disjoint 4-cocircuits, and M restricted to either pair is isomorphic to $M(K_{2,4})$ by Proposition 4.6. Let $E(M) = \{x_1, x_2, \ldots, x_{12}\}$; then, without loss of generality, M has cocircuits $D_1 =$

$\{x_1, x_2, x_3, x_4\}$, $D_2 = \{x_5, x_6, x_7, x_8\}$, $D_3 = \{x_1, x_5, x_9, x_{10}\}$, and $D_4 = \{x_2, x_6, x_{11}, x_{12}\}$. Without loss of generality, the circuits contained in $M|(D_1 \cup D_2)$ and $M|(D_3 \cup D_4)$ are given by Figure 4.25.

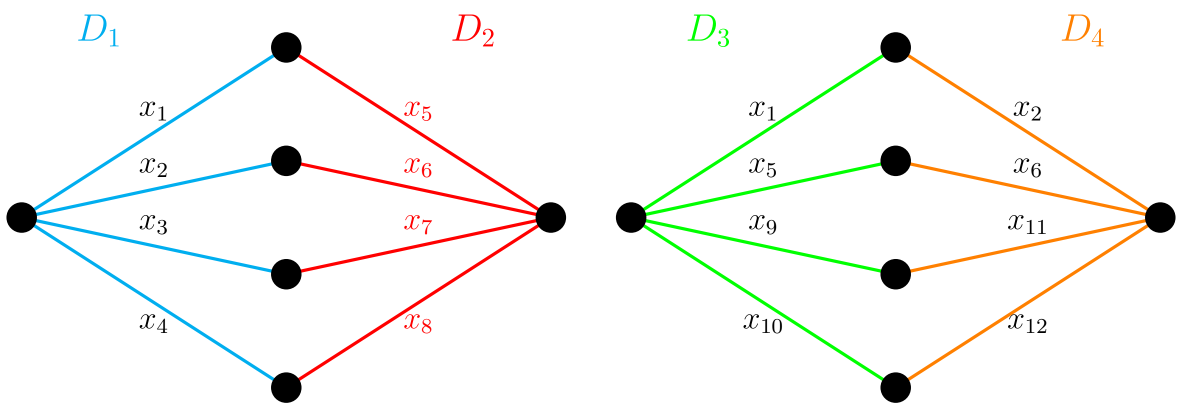

Figure 4.25: The underlying $K_{2,4}$ structure in $M|(D_1 \cup D_2)$ and $M|(D_3 \cup D_4)$.

We first prove that

Claim 4.35.1. *$r(M) = 5$.*

Consider the set $S = E(M) - \{x_3, x_4, x_7, x_8\}$. As $\{x_3, x_4, x_7, x_8\}$ is a circuit, S is a cohyperplane. Further, $\mathrm{cl}(\{x_1, x_2, x_5, x_9, x_{10}\}) = S$, so $r(S) \leq 5$. Since M is 4-connected, $3 \leq \lambda_M(S) = r(S) + r^*(S) - |S| \leq 5 + r^*(S) - 8$. Therefore, $r^*(S) \geq 6$ and $r(M^*) \geq 7$, so $r(M) \leq 5$. Clearly $r(M) \geq 4$, so we must only prove $r(M) \neq 4$.

If $r(M) = 4$, then $B = \{x_3, x_7, x_9, x_{11}\}$ is a basis, as it cannot be a circuit by orthogonality. Consider the fundamental circuit $C(x_1, B)$. Neither x_7 nor x_{11} may be in $C(x_1, B)$, as there are no other elements from D_2 or D_4. Therefore, $|C(x_1, B)| \leq 3$, a contradiction. Thus $r(M) = 5$.

There are eight elements that appear in only one of the given 4-cocircuits; namely, the members of $X = \{x_3, x_4, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}\}$. Consider a 5-element subset of those elements that meets every given 4-cocircuit at least once. Note that such a subset must contain two elements from one 4-cocircuit, and one element from each other 4-cocircuit. Therefore, such a set must be a basis, by orthogonality. Every fundamental circuit that is given by one of these bases together with an element e from $E(M) - X$ must either be a 4-circuit or a 5-circuit. The number of elements in the fundamental circuit depends on whether e is contained in the same 4-cocircuit from which the basis has two elements; if so, then we get a 4-circuit, and if not, then it must be a 5-circuit. For example, if we choose $B' = \{x_3, x_4, x_7, x_9, x_{11}\} \subseteq X$ to be our basis, then $C(x_1, B') = \{x_1, x_3, x_4, x_9\}$, while $C(x_5, B') = \{x_3, x_4, x_5, x_7, x_11\}$. It should be noted that some of the 4-circuits determined in this way intersect in 3 elements. Therefore, we get the following 5-point planes in M: $\{x_1, x_3, x_4, x_9, x_{10}\}$, $\{x_5, x_7, x_8, x_9, x_{10}\}$, $\{x_2, x_3, x_4, x_{11}, x_{12}\}$, and $\{x_6, x_7, x_8, x_{11}, x_{12}\}$. We call the 12-element matroid with these 4-circuits M_{12}, and proceed to prove its uniqueness.

Suppose there is some other 12-element matroid, say M', with every element in a 4-cocircuit and every pair of elements in a 4-circuit, and let M' share ground sets with M_{12}. The circuits and cocircuits mentioned above are forced, so M' and M_{12} agree on those. As $M' \ncong M_{12}$, there must be a minimal subset, T, such that T is independent in one and dependent in the other. Since $r(M') = r(M_{12}) = 5$, it must be that $4 \leq |T| \leq 5$. If $|T| = 4$, then it must be independent in M_{12}. The only 4-element independent sets in M_{12} meet at least one 4-cocircuit in a single element. Therefore, $|T| = 5$.

In this case, T is a circuit in one of M' or M_{12}, and a basis in the other. A 5-circuit cannot be a subset of $E(M) - X$, so, without loss of generality, $x_1 \in T$. We prove next that

Claim 4.35.2. *$x_2 \in T$.*

Suppose not. We may further assume that $x_5 \notin T$, as the permutation $(x_2, x_5)(x_3, x_9)(x_4, x_{10})$ is an automorphism of M. Now, T must contain one element from each $\{x_3, x_4\}$ and $\{x_9, x_{10}\}$, in order to avoid an orthogonality contradiction with D_1 or D_3. These pairs are symmetric under the automorphism given by the permutation $(x_3, x_4)(x_7, x_8)(x_9, x_{10})(x_{11}, x_{12})$, so, without loss of generality, $\{x_1, x_3, x_9\} \subseteq T$. As T is a basis in one matroid, it must meet every cocircuit of that matroid, and, as it is a circuit in the other matroid, it must do so in at least two elements. However, it does not yet have an element from either D_2 or D_4, and there are only two undetermined elements; a contradiction.

Therefore, $x_2 \in T$. By similar reasoning, T must also contain one of x_5 or x_6. These elements are symmetric under the permutation $(x_1, x_2)(x_5, x_6)(x_9, x_{11})(x_{10}, x_{12})$, so we may assume $x_5 \in T$. In order to avoid an orthogonality contradiction with one of D_2 and D_4, an element from each $\{x_6, x_7, x_8\}$ and $\{x_6, x_{11}, x_{12}\}$ must be in T. This element cannot be x_6, as $\{x_1, x_2, x_5, x_6\}$ is a circuit. Therefore, without loss of generality, $T = \{x_1, x_2, x_5, x_7, x_{11}\}$. But, by circuit elimination, there is a circuit $C \subseteq (T \cup \{x_1, x_2, x_5, x_6\}) - \{x_2\} = \{x_1, x_5, x_6, x_7, x_{11}\}$. Clearly $x_1 \notin C$, as otherwise $|C \cap D_1| = 1$, but then $x_5 \notin C$, otherwise $|C \cap D_3| = 1$. Therefore $|C| \leq 3$, a contradiction. Thus no such T exists, and $M' \cong M_{12}$. Thus $M_{12} \cong M(P)$.

□

Proposition 4.36. *Let $|E(M)| = 14$. Then M is a 4-connected matroid in which every element is in a 4-cocircuit and every pair of elements in a 4-circuit if and only if $M \cong M_{14}$.*

Proof. Clearly, $r(M(Q)) = 6$. It is straightforward to verify that $M(Q)$ is 4-connected and has property $(P2)$.

Now, suppose that M satisfies the given conditions. By Proposition 4.19, M has three pairwise-disjoint 4-cocircuits D_1, D_2, and D_3. By Lemma 4.17, we know that $M|(D_1 \cup D_2 \cup D_3) \cong M(K_{3,4})$. Let $E(M) = \{x_1, x_2, \ldots, x_{14}\}$. Without loss of generality, we may assume that $D_1 = \{x_1, x_2, x_3, x_4\}$, $D_2 = \{x_5, x_6, x_7, x_8\}$, and $D_3 = \{x_9, x_{10}, x_{11}, x_{12}\}$, and the circuits contained in $D_1 \cup D_2 \cup D_3$ are given by Figure 4.22.

Therefore, without loss of generality, we may assume there are 4-cocircuits containing x_{13} and x_{14} which are given by $D_4 = \{x_1, x_5, x_9, x_{13}\}$ and $D_5 = \{x_2, x_6, x_{10}, x_{14}\}$, respectively. As D_4 and D_5 are disjoint, $M|(D_4 \cup D_5) \cong M(K_{2,4})$, and, in order to avoid violating orthogonality, the pairs $\{x_1, x_2\}$, $\{x_5, x_6\}$, $\{x_9, x_{10}\}$, and $\{x_{13}, x_{14}\}$ always appear together in the 4-circuits contained in $D_4 \cup D_5$, as depicted in Figure 4.26.

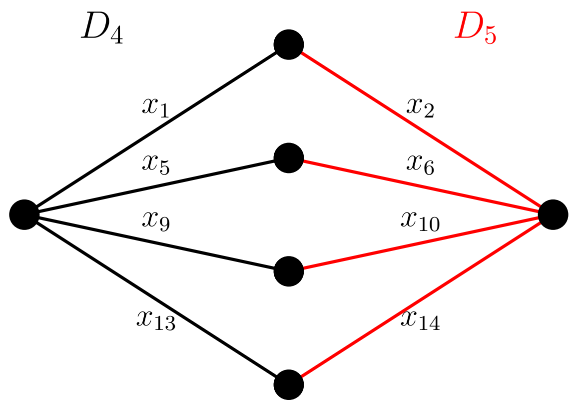

Figure 4.26: This $K_{2,4}$ determines the circuits in $M|(D_4 \cup D_5)$.

We prove that

Claim 4.36.1. *$r(M) = 6$.*

Consider $X = (E(M) - \{x_3, x_4, x_7, x_8\})$. Observe that $\mathrm{cl}(\{x_1, x_2, x_5, x_9, x_{11}, x_{13}\}) = X$, so $r(X) \leq 6$. As M is 4-connected, $3 \leq \lambda_M(X) = r(X) + r^*(X) - |X| \leq r^*(X) - 4$, so $r^*(X) \geq 7$. As X is a cohyperplane, this implies $r^*(M) \geq 8$, so $r(M) \leq 6$. Now consider a set, Y, with one element from each 4-cocircuit, such that none of the elements is in more than one 4-cocircuit. Clearly, $|Y| = 5$, and Y is independent. Therefore $r(M) \not\leq 4$. If $r(M) = 5$, then Y is a basis of M. In this case, consider the fundamental circuit $C(x_1, Y)$. Such a circuit must not have elements from D_2, D_3, or D_5, otherwise it will violate orthogonality. Therefore $|C(x_1, Y)| \leq 3$, a contradiction. Thus $r(M) = 6$.

We now prove that this structure allows for exactly one matroid on 14 elements. Before we begin, note the following six 4-circuits that are, without loss of generality, necessarily in any 14-element matroid having property $(P2)$: $\{x_i, x_{i+2}, x_{i+3}, x_{13}\}$ and $\{x_j, x_{j+1}, x_{j+2}, x_{14}\}$ for each $i \in \{1, 5, 9\}$ and $j \in \{2, 6, 10\}$. To see that these must exist, consider a 4-circuit C' containing $\{x_{i+2}, x_{13}\}$, and let $D' \in \{D_1, D_2, D_3\}$ be the 4-cocircuit that contains x_{i+2}. In order to avoid an orthogonality contradiction with D_4 or D', we know $x_1 \in C'$. Then, the last element of C' must also come from D', and it cannot be x_j by orthogonality. Therefore, $C' = \{x_i, x_{i+2}, x_{i+3}, x_{13}\}$. The other case holds similarly: simply swap x_1 with x_2, and replace

$(x_i, x_{i+2}, x_{i+3}, x_{13})$ by $(x_j, x_{j+1}, x_{j+2}, x_{14})$ in the above argument. With this in mind, suppose there are two such matroids, and call them M_{14} and M'. If $M_{14} \ncong M'$, then there is a minimal set T that is independent in one and dependent in the other. Therefore, $4 \leq |T| \leq 6$. We treat each possibility in a separate case.

Case 4.36.2. *Suppose* $|T| = 4$.

As T is a circuit in one matroid, it must contain at at least two elements from each 4-cocircuit it meets. Suppose one of $\{x_1, x_2, x_5, x_6, x_9, x_{10}\}$ is not in T. Then neither x_{13} nor x_{14} are in T, and so $T \subseteq \{x_3, x_4, x_7, x_8, x_{11}, x_{12}\}$. However, we have accounted for all such 4-circuits in both matroids. Therefore, without loss of generality, $x_1 \in T$. In this case, T must also contain one element from each $\{x_2, x_3, x_4\}$ and $\{x_5, x_9, x_{13}\}$. We can prove that

Claim 4.36.2.1. $x_{13} \notin T$.

Suppose $x_{13} \in T$. If $x_2 \in T$ as well, then $T = \{x_1, x_2, x_{13}, x_{14}\}$, which is a circuit in both M_{14} and M'. Therefore, $x_2 \notin T$, in which case $T = \{x_1, x_3, x_4, x_{13}\}$. As noted above, this is a circuit in both matroids.

Therefore, $x_{13} \notin T$, and, without loss of generality, $x_5 \in T$. In this case, T must contain an element from each of $\{x_2, x_3, x_4\}$ and $\{x_6, x_7, x_8\}$, but then $T \subseteq D_1 \cup D_2$, and all such 4-circuits are forced. Thus $|T| \neq 4$.

Case 4.36.3. *Suppose* $|T| = 5$.

Again, T is a circuit in one matroid, and we may assume $x_1 \in T$.

Subcase 4.36.3.1. *Suppose* $x_5 \in T$.

In this case, T must contain an element from both $\{x_2, x_3, x_4\}$ and $\{x_6, x_7, x_8\}$ in order to avoid an orthogonality contradiction in whichever matroid T is a circuit.

We show next that x_2 is not in T. Assume the contrary. Then T also contains an element from $\{x_6, x_{10}, x_{14}\}$. This element cannot be x_6, as $\{x_1, x_2, x_5, x_6\}$ is a circuit in both matroids.

Further, $x_{10} \notin T$, as then T must contain an element from $\{x_9, x_{11}, x_{12}\}$. Then, without loss of generality, $T = \{x_1, x_2, x_5, x_7, x_{14}\}$. In whichever matroid T is a circuit, the rank of T is 4. However, in both matroids $\mathrm{cl}(T) = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_{13}, x_{14}\}$, which is a hyperplane. As both matroids have rank 6, neither may have a rank-4 hyperplane, a contradiction.

We deduce that x_2 is not in T, and, by symmetry, x_6 is not in T. Hence, without loss of generality, we may assume $x_3 \in T$. As $\{x_1, x_3, x_5, x_7\}$ is a circuit in both matroids, $x_7 \notin T$, and so $x_8 \in T$. The fifth element of T cannot come come from D_3 or D_5, and $x_4 \notin T$ as $\{x_1, x_4, x_5, x_8\}$ is a circuit in both matroids, so $T = \{x_1, x_3, x_5, x_8, x_{13}\}$. As T is independent in one matroid, $\mathrm{cl}(T)$ is a hyperplane of that matroid. However, $T \subseteq ((E(M_{14})-D_3) \cap (E(M_{14})-D_5))$; that is, T is contained in the intersection of two hyperplanes. Thus $r(T) \leq 4$, a contradiction. We now know that

Subcase 4.36.3.2. $x_5 \notin T$

By symmetry, we also have $x_9 \notin T$. In order to avoid an orthogonality contradiction with D_4, this implies $x_{13} \in T$. There must be an element from $\{x_2, x_3, x_4\}$ in T.

We show next that x_2 is not in T. Assume the contrary. Then T needs an element from $\{x_6, x_{10}, x_{14}\}$. Suppose $x_6 \in T$. In this case, T must contain an element from $\{x_5, x_7, x_8\}$, and, since x_7 and x_8 are symmetric, we may assume $T = \{x_1, x_2, x_6, x_7, x_{13}\}$. However, this is symmetric to the previous case in which T was equal to $\{x_1, x_2, x_5, x_7, x_{14}\}$, under the automorphism given by the permutation $(x_1, x_2)(x_5, x_6)(x_9, x_{10})(x_{13}, x_{14})$. Therefore, if $x_2 \in T$, then $x_6 \notin T$, and, by symmetry, neither is x_{10}. It must be that $x_{14} \in T$. But then $\{x_1, x_2, x_{13}, x_{14}\}$, a circuit on both matroids, is a subset of T, a contradiction. We conclude that x_2 is not in T.

This implies, without loss of generality, that $x_3 \in T$. Now, neither x_6 nor x_{10} may be in T, as they are each contained in two 4-cocircuits disjoint from T. Similarly, $x_{14} \notin T$, as then one of x_6 or x_{10} would be forced in order to avoid an orthogonality contradiction with D_5. Therefore, the remaining members of T must be either $\{x_7, x_8\}$ or $\{x_{11}, x_{12}\}$. These are symmetric under the automorphism given by the permutation $(x_5, x_9)(x_6, x_{10})(x_7, x_{11})(x_8, x_{12})$, so it suffices to assume $T = \{x_1, x_3, x_7, x_8, x_{13}\}$. However, as both M_{14} and M' contain circuits $\{x_1, x_3, x_5, x_7\}$ and $\{x_5, x_7, x_8, x_{13}\}$, we may see, by circuit elimination on these excluding x_5, that this T is a circuit in both matroids. This last contradiction proves that $|T| \neq 5$.

We are left with one possibility.

Case 4.36.4. *Suppose* $|T| = 6$.

In this case, T is a circuit in one of M_{14} and M', and a basis in the other. By similar reasoning to the previous cases, we may assume $x_1 \in T$. We will first show that

Claim 4.36.4.1. $T \cap \{x_5, x_9\} \neq \emptyset$.

If neither x_5 nor x_9 is in T, then T must contain x_{13}. Furthermore, T must contain a second element from D_1. As T is a basis in one matroid, it must meet all the cocircuits of that matroid; specifically, it must meet both D_2 and D_3. However, as T is a circuit in the other matroid, it must meet both D_2 and D_3 in at least two elements. Since D_2 and D_3 are disjoint, and there are only three unaccounted-for elements of T in this case, this is a contradiction.

Therefore, T must contain one of x_5 and x_9. Without loss of generality, say $x_5 \in T$. Now T needs a second element from each D_1 and D_2.

Claim 4.36.4.2. $T \cap \{x_2, x_6\} \neq \emptyset$.

If neither x_2 nor x_6 is in T, then, without loss of generality, $\{x_1, x_3, x_5, x_8\} \subseteq T$. In this case, T must still meet both D_3 and D_5 in at least two elements, and $|D_3 \cap D_5| = 1$, a contradiction.

Now T must contain one of x_2 and x_6. These are symmetric under the automorphism given by the permutation $(x_1, x_5)(x_2, x_6)(x_3, x_7)(x_4, x_8)$, so we may assume that $x_2 \in T$. As $\{x_1, x_2, x_5, x_6\}$ is a circuit in both matroids, we may further assume that $x_7 \in T$, in order to avoid an orthogonality contradiction with D_2. Now T must have a second element from D_5, and must meet D_3 in two elements. This forces $x_{10} \in T$, and the final element of T is either x_{11} or x_{12}. If $T = \{x_1, x_2, x_5, x_7, x_{10}, x_{11}\}$, then consider that the sets $\{x_1, x_2, x_5, x_6\}$ and $\{x_6, x_7, x_{10}, x_{11}\}$ are circuits in both matroids. Therefore $T = (\{x_1, x_2, x_5, x_6\} \cup \{x_6, x_7, x_{10}, x_{11}\}) - \{x_6\}$ is a circuit in both matroids, a contradiction. This brings us to the final possibility for T in this case: $T = \{x_1, x_2, x_5, x_7, x_{10}, x_{12}\}$. By inspection of its closure, we see that $\mathrm{cl}(T)$ is the entire matroid in both cases. Thus T is a basis in both matroids, and M_{14} is unique. Thus $M_{14} \cong M(Q)$. $\square$

4.7 The Main Result

This section will conclude our analysis of matroids having property $(P2)$. Once the size of M is at least 16, the matroids that satisfy our conditions all fall into one family; that is, $M \cong M(K_{4,n})$ for some $n \geq 4$. The proof of this is a straightforward application of induction on the number of elements, requiring two quick preceding lemmas.

Lemma 4.37. *Let $M|X \cong M(K_{2,4})$ and D be a 4-cocircuit of M meeting X. Then either D contains exactly one element from each of the four series pairs in $M|X$, or D meets X in a series pair of $M|X$.*

Proof. Suppose not. Let $\{x_i, y_i\}$ for $i \in \{1, 2, 3, 4\}$ be the series pairs of $M|X$. Since M is 4-connected, $D \cap X \neq \{x_i, y_i, x_j, y_j\}$ for every $\{i, j\} \subseteq \{1, 2, 3, 4\}$. Therefore, D must meet some series pair in exactly one element, and another series pair not at all. Without loss of generality, $D \cap \{x_1, y_2\} = \{x_1\}$ and $D \cap \{x_2, y_2\} = \emptyset$. But $\{x_1, x_2, y_1, y_2\}$ is a circuit. This contradiction completes the proof of the lemma. $\square$

Lemma 4.38. *If $|E(M)| \geq 16$, then M has four pairwise-disjoint 4-cocircuits. Further, $|E(M)| = 4n$ for some $n \geq 4$, and M may be partitioned into 4-cocircuits.*

Proof. By Proposition 4.19, M has three pairwise-disjoint 4-cocircuits, D_1, D_2, and D_3, forming a local $K_{3,4}$ structure. Let the elements in these sets be $\{x_1, x_2, x_3, x_4\}$, $\{x_5, x_6, x_7, x_8\}$, and $\{x_9, x_{10}, x_{11}, x_{12}\}$, respectively. Then, the circuits contained in $D_1 \cup D_2$, $D_1 \cup D_3$, and $D_2 \cup D_3$ are given by Figure 4.22.

Let x_{13}, x_{14}, x_{15}, and x_{16} be distinct elements of $E(M) - (D_1 \cup D_2 \cup D_3)$. Each of these elements is in a 4-cocircuit. We may assume that each of these 4-cocircuits contains at least one element from $D_1 \cup D_2 \cup D_3$. As $M|(D_1 \cup D_2 \cup D_3) \cong M(K_{3,4})$, orthogonality forces each of these 4-cocircuits to contain three elements of $D_1 \cup D_2 \cup D_3$. Moreover, by Lemma 4.37, these three elements form a triad in $M|(D_1 \cup D_2 \cup D_3)$. It follows that two such 4-cocircuits are disjoint, otherwise they are forced to share three elements, a contradiction to Proposition 4.2. Therefore, M has four disjoint 4-cocircuits.

It is clear, then, that when $|E(M)| = 16$, there is a partition of $E(M)$ into 4-cocircuits. Suppose $|E(M)| > 16$, and that M cannot be partitioned into 4-cocircuits. Let $\{D_1, D_2, \ldots, D_k\}$ be a maximum-sized set of pairwise-disjoint 4-cocircuits of M. Then $k \geq 4$. Let e be an element of $E(M) - (D_1 \cup D_2 \cup \cdots \cup D_k)$ and D be a 4-cocircuit containing e. Then $D \cap (D_1 \cup D_2 \cup D_3 \cup D_4) \neq \emptyset$. But, by Lemma 4.37, D must contain at least four elements from $D_1 \cup D_2 \cup D_3 \cup D_4$. This contradiction completes the proof of the lemma. $\square$

Proposition 4.39. *If* $|E(M)| \geq 16$, *then* $M \cong M(K_{4,n})$ *for some* $n \geq 4$.

Proof. We argue by induction on n. Suppose $n = 4$ and let D_1, D_2, D_3, and D_4 be pairwise-disjoint 4-cocircuits of M. By Proposition 4.19, the restriction of M to any three of these is isomorphic to $M(K_{3,4})$. Therefore, if $\phi : E(M(K_{4,4})) \to E(M)$ is a bijection that maps the 4-circuits and 4-cocircuits of $M(K_{4,4})$ to those of M, then ϕ is a weak map. By Theorem 3.6, this means $M(K_{4,4}) \cong M$.

Now, suppose that $|E(M)| = 4i$ implies $M \cong M(K_{4,i})$ for all $4 \leq i \leq m - 1$, and consider M such that $|E(M)| = 4m$. If $M \ncong M(K_{4,m})$, then there is a minimal set Z that is independent in one of these matroids and is a circuit in the other. As such, if Z has one element from a 4-cocircuit of M, then it has at least two. Suppose $Z \cap D = \emptyset$ for some 4-cocircuit D in the cocircuit partition of $E(M)$. Then $Z \subseteq M \backslash D$. But $M \backslash D$ has property $(P2)$ and $|E(M \backslash D)| \geq 16$, so, by the induction hypothesis, Z must meet each of the 4-cocircuits that partitions M. As there are m 4-cocircuits in the partition, we have $|Z| \geq 2m$. Also note that $r(M) = r(M \backslash D) + 1 = r(M(K_{4,m-1})) + 1 = m + 3$. Since Z is assumed to be a circuit in one M or $M(K_{4,m})$, we get $2m \leq |Z| \leq m + 4$. This inequality fails if $m > 4$. Thus the proposition follows by induction. $\square$

Putting it all together, we get the main theorem of this chapter.

Theorem 4.40. *Suppose M is a 4-connected matroid. If M has every element in a 4-cocircuit and every pair of elements in a 4-circuit, then M is one of the following matroids:* $U_{3,6}$, $M_{8,1}$, $M_{8,2}$, $M_{8,3}$, $M_{8,3^+}$, $M_{8,4}$, $M_{8,4^+}$, $M_{8,5}$, $M_{8,6}$, $M_{8,7}$, $M_{8,7^+}$, $M_{8,8a}$, $M_{8,8b}$, $M_{8,9a}$, $M_{8,9b}$, $M_{8,9b^+}$, $M_{8,10}$, $M_{8,10^+}$, $M_{8,10^{++}}$, $M_{8,11}$, $M_{8,12}$, F_7^+,$M_{9,1}$, $M_{9,1a}$, $M_{9,1b}$, $M_{9,2}$, $M_{9,3}$, $M_{9,3^+}$, $M_{9,4}$, $M_{9,4^+}$, $M_{9,5}$, $M_{9,6}$, R_{10}, M_{12}, M_{14} *or* $M(K_{4,n})$ *for some* $n \geq 4$.

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Vita

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